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AN ESSAY ON THE MATHEMATICAL THEORY OF FREEDOM: GABOR AND GABOR (1954) REVISITED

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An essay on the mathematical theory of freedom: Gabor and Gabor (1954) revisited*

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Abstract

In this paper we review an essay , written half a century ago by D. Gabor and A. Gabor (1954), concerned with the definition and measurement of social freedom. D and A. Gabor identify two crucial aspects of freedom, freedom as diversity and freedom as independence, and propose a general framework which can be used for a rigorous quantitative discussion. The approach they propose is objective, because it is based on observable actions rather than individual preferences and ex-ante opportunities for choice; and statistical, in that it relates to large population rather than to single individuals. In this paper we review G. & G. 's essay, assess the merits of their approach, discuss, modify and clarify their analitycal framework, and study the links with the more recent literature on the measurement of freedom.

1 Introduction

In this paper we review and critically assess an essay, written half a century ago by D. and A. Gabor, on "the mathematical theory of freedom".

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This old essay is concerned with the definition and measurement of freedom. Specifically, they deal with the concept of social freedom. What do we mean by a "free society"? How free is a society? Can we say that society A exhibits a higher degree of freedom than society B? Can we rank societies on the basis of freedom? These are the kind of questions that motivate their work.

We believe these questions are still relevant. In fact, while the notion of freedom is given a central place in contemporary philosophical and political debates, it is surprising that not much effort has been devoted, until now, to the construction of a rigorous metric for comparing societies in terms of freedom.

The measures used by many organizations, among them the United Nations, in order to quantify the freedom of different countries can and have been legitimately criticized as over-crude, with unclear theoretical presuppositions.

On the other hand, the problem of measuring freedom has been addressed in a rigorous way recently by normative economists. However, this literature focuses only on the individual level; moreover, in our view, this literature is too abstract and, as a consequence, no empirical work has followed from it.

We believe that Gabor and Gabor's (1954) (henceforth G-G) approach, which is based on sound theoretical grounds, is actually useful for operational purposes, having identified some crucial aspects of freedom and proposed a general framework which can be used for a rigorous quantitative discussion. The approach proposed by G-G is objective, because it is based on observable actions rather than individual preferences or ex-ante opportunities for choice; and statistical, in that it relates to large population rather than to single individuals.

Our mid-term program is that of building on this old essay in order to provide a rigorous metric of social freedom. However, we leave for a future companion paper the full exploitation of the framework they propose. In this essay, more modestly, we review G-G 's paper, assess the merits of their approach, discuss - and modify slightly - their analytical framework, and study the links with the more recent literature on the measurement of freedom.

2 Defining freedom

Freedom, according to G-G, has two main features: variety and independence. A measure of freedom should capture both components.

Following MacCallum (1967), a general description of liberty takes the form of a triadic relationship: an agent (individual or collective) is free from a given set of constraints to choose among a given set of options. A definition of freedom, and a measure consistent with it, is therefore obtained by specifying the set of options, the constraints and the agent they are referred to.

Thus the first aspect of freedom concerns the options people may choose: what are the actions open to an individual in a given situation? Given a set of options, trough the act of choice only one of these options will materialize, depending on the desires and preferences of the agent. Therefore, looking only at the choices - i.e., adopting an objective approach - does not tell us anything about the ex-ante available options. However, this is only true at the individual level. If the individual is a member of a large population then an objective approach may be legitimate: it may be sensible to say that the real options individuals are free to choose in a society must manifest themselves in the actions or choices of at least a fraction of the population.

> We take the view that what individuals "care to do" or "may care to do" must manifest itself in the actions of at least a fraction of the population. If there is no regulation which forbids bank clerks to wear boiler suits, yet among a thousand bank clerks on a thousand mornings there is not one who appears at work in a boiler suit, we conclude that they are *de facto* not free to wear that garment. (G-G, p.332)

With this methodology one uses information about ex-post choices in a society to infer the ex-ante opportunities of an individual in that society[1]. The approach is phenomenological rather than procedural and it is statistical rather than individual. Hence, G-G essentially propose to measure social freedom by measuring the freedom of a representative agent in the relevant society.

This phenomenological approach could be criticized on the basis of the following argument. Using information about ex-post choices instead of potential choices one basically overlooks the conditions in which decisions are

[1]More recently, this approach has been defended and used by Suppes (1995).

made. But, it could be argued, the conditions under which decisions are made are relevant for the assessment of freedom; for, a society in which there is a plurality of accessible options, but where, for cultural reasons, only few of these options are chosen, is a free society. In other words, the pluralism should be assessed in the ex-ante distribution of opportunity for choice; not in the ex post distribution of choices.

Certainly there is something in this criticism. However, to appreciate the plurality of options in a society by looking at the ex-ante options, one should have detailed informations about the process of choice. For, to make some options which are only formally available genuine opportunities open to individuals it is necessary that the act of choosing is not restricted by external pressures or influences. But it is extremely difficult to have quantitative data about the ex-ante options and process leading to the final choices; on the contrary, excellent data can be obtained on the choices.

Hence, roughly speaking, the more the options chosen by at least some member of the society, the higher the degree of freedom individuals in that society enjoy. Thus freedom manifests itself in the plurality of the chosen options; and it is measured as statistical spread of choices.

The diversity aspect of freedom is only one of its features. A second aspect, G-G say, is the independence aspect. We can grasp the meaning and the role of this aspect of freedom by considering the constraints an individual faces in the context of a choice. From this viewpoint the question is: when is an agent free from "external circumstances" to do this and that, to chose this and that? Which factors or conditions influence her ability to choose a given option? This question evokes a problem of independence of the available options from external factors. Given a set of options, an individual is free, in the current sense, if the set of options open to her is independent from external circumstances. This is the independence aspect considered by G-G. Again, the approach to measurement they propose is statistical: the degree of independence is to be evaluated by looking at a large population: an individual is free from factor x to choose option y if, in the relevant society, we observe statistical independence of y and x.

This solution leaves open the question of which external factors are relevant when measuring freedom. In fact, my set of options may be dependent from a plurality of factors: natural factors, social conditions, my own past choices and actions. Of which of these classes must one be independent to be free?

A guide in this search can be found in the discussion on the constraints

which are relevant for the definition of freedom (see Berlin 1958, Bobbio 1956, Miller 1983, Oppenheim 1985) . Recalling MacCallum scheme, a conception of freedom must define three things: an agent, X, who is free from constraints, Y, to do an action (or to be something), Z. All disagreements about the definition of freedom therefore boil down to questions about the extension of factors X, Y and Z. Specifically relevant for the present discussion is the debate over whether the Y factor should indicate merely "external" or also "internal" constraints, and over whether Y includes only constraints imposed by other agents or also those imposed by nature. Does the poverty of a tramp who is unable to dine at the Ritz restrict his freedom?

One plausible solution is that of arguing that impersonal economic forces cannot reasonably be interpreted as constraints on freedom. At this point, the "Y factor" in MacCallum's formula gets narrowed down to only those obstacles that are imposed deliberately, or that are at least foreseeable, by other individuals. But an alternative answer would be that he is unfree only if the fact that he is poor - so the constraints that prevent him to dine at the Ritz - cannot be attributed to his choice, nor to fortune, but to an unjust distribution of resources in the society, so that we can attribute to the society the will that causally determines the situation of poverty.

Using a similar argument, we can argue that "the conditions from which one must be independent to be free" are to be identified by looking at "the dependences that are considered unjust in the given society". This consideration links the theory of freedom to the theory of justice.

In the rest of the paper we modify slightly the analytical framework used by G-G and follow their approach in proposing numerical measures of freedom and of its two constituent elements, plurality and independence.

3 Setting

3.1 Choices

Let $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{z}$ represent *classes of choices*. For example, $\mathbf{a}$ may represent the choice of newspaper, $\mathbf{b}$ the choice of occupation, etc.

Within each class, there is a number of options (for example, each available newspaper, any possible combination of them, and the choice of not reading any). Thus each class of choices can be represented as a set of options $\mathbf{a} = \{a_1, a_2, \cdots, a_{|\mathbf{a}|}\}$, $\mathbf{b} = \{b_1, b_2, \cdots, b_{|\mathbf{b}|}\}$, $\cdots$, $\mathbf{z} = \{z_1, z_2, \cdots, z_{|\mathbf{z}|}\}$

where, for a set S, $\mid S \mid$ denotes the cardinality of S.

Let X denote the Cartesian product of the sets of choice options, $X = \mathbf{a} \times \mathbf{b} \times \cdots \times \mathbf{z}$. Note that $\mid X \mid$ is equal to $\mid \mathbf{a} \mid \cdot \mid \mathbf{b} \mid \cdots \mid \mathbf{z} \mid = N$, say. We then index each possible choice bundle by $i = 1, \cdots, N$, by ordering lexicographically each possible combination of the options in each choice set. A typical choice bundle will be denoted as $x_i \in X$. Thus, for example, if $\mathbf{a}$ contains $\mid \mathbf{a} \mid$ elements, then for $i = 1, \cdots, N/(\mid \mathbf{a} \mid)$ the choice bundle x_i will be such that the choice a_1 is made, for $i = N/(\mid \mathbf{a} \mid) + 1, \cdots, 2N/(\mid \mathbf{a} \mid)$ x_i will be such that the choice a_2 is made, and so on.

3.2 Individual Characteristics

Let $\alpha, \beta, \cdots, \omega$ represent *classes of individufl characteristics.* For example, α may represent race, β ability, etc.

Within each class, there is a number of elements (for example, different degrees of ability). Thus each class of characteristics can be represented as a set of elements $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_{|\alpha|}\}$, $\beta = \{\beta_1, \beta_2, \cdots, \beta_{|\beta|}\}$, $\cdots$, $\omega = \{\omega_1, \omega_2, \cdots, \omega_{\ |\omega|}\}$.

Let Y denote the Cartesian product of the sets of individual characteristics, $Y = \alpha \times \beta \times \cdots \times \omega$. Let $\mid Y \mid = M$, say. We then index each possible individual characterization by $j = 1, \cdots, M$, by ordering lexicographically each possible combination of the options in each individual characteristics set. A typical individual characterization will be denoted as $y_j \in Y$. Thus, for example, if α contains $\mid \alpha \mid$ elements, then for $j = 1, \cdots, N/(\mid \alpha \mid)$ the individual characterization y_j will be such that the individual will have characteristic α_1; for $j = N/(\mid \alpha \mid) + 1, \cdots, 2N/(\mid \alpha \mid)$, y_j will be such that the individual will have characteristic α_2, and so on.

3.3 The society

All the information about this society is contained in a bivariate contingency table P with typical element p_{ij}, $i = 1, \cdots, N$, $j = 1, \cdots, M$ which denotes the proportion of individuals in the society which have chosen the bundle x_i and have individual characterization y_j. We denote the marginal distributions of the choice bundles by $(p_{1+}, p_{2+}, \cdots, p_{N+})$ and of the individual characterizations by $(p_{+1}, p_{+2}, \cdots, p_{+M})$. Let $D_{N,M}$ be the set of $N \times M$

contingency tables, that is,

$$D_{N,M} = \left\{ p_{ij} : p_{ij} \geq 0, N \geq i \geq 1, M \geq j \geq 1, \sum \sum p_{ij} = 1 \right\}.$$

We want to rank members of $D_{N,M}$ (''societies") by their amount of freedom.

4 Measuring freedom

4.1 Freedom as diversity

G-G propose to evaluate the amount of diversity freedom in our society by considering the marginal distribution of the choice bundles $(p_{1+}, p_{2+}, \cdots, p_{N+})$. An index of freedom as diversity is then a real valued function from the N-dimensional simplex. Let us call this function H. G-G use a modified form of Shannon's entropy measure (1949), cast in freedom terms:

$$D = -\frac{1}{\sum_{i=1}^{N} \frac{p_{i+}}{\log p_{i+}}}.$$

Note that this measure has been recently discussed and employed by Suppes (1995), apparently unaware of G-G earlier contribution.

Now, Shannon's entropy measure hass been much discussed in the literature on information theory, and various axiomatizations exists in the literature (see e.g. Shannon, 1948 and Lee, 1963). Looking at the axioms that characterize the entropy measure is a good exercise to understand its suitability as a measure of freedom as diversity (To the discussant: this will be briefly mentioned in the presentation). In particular, we will argue that it has the same kind of flavour than using cardinality to measure individual freedom of choice (see e.g. Pattanaik and Xu (1990) and Barbera' et al. (2001) for a survey).

Now, it can be argued that equating freedom with diversity blurs tye distinction between choices that are made under external constraints and choices which are freely made. For example, both G-G and Suppes (1995) use the example of the diversity of votes going to different political parties in an election as a measure of the freedom of this election. Or, if in the choice of profession we observe little diversity, it is important to know if the lack of diversity is due to external constraints (limitations to individual freedom) or to free choice of the citizens.

A more appropriate measure of freedom which looks in this direction could be then modeled in the following fashion: let q be the vector of proportions that individuals would choose without external constraints, and let p be the actual vector of proportions(the marginal distribution as above). Then the lack of liberty is a function of the *distance* between p and q in an appropriate metric. There are many distance functions: we hope to characterize one in a future paper. If we have data on q and p, this approach could be empirically impleented. One possible example is about voting at political elections: if the researcher has data on voters' intentions and actual voting data, by choosing an appropriate distance function we can quantify the (lack of) freedom in this society.

4.2 Freedom as independence

How to implement a measure of freedom as independence?

To capture this aspect of freedom, G-G crucially use information about individual characteristics. Now, suppose we partition the population into equivalence classes, identified by the relevant characteristics, and assume that these classes are big enough to be statistically significant. Then we can say that the real options individuals in a given class are free to choose must manifest themselves in the actions or choices of at least a fraction of the population in that class.

Suppose there is an option, say x, that in a given class i does not manifest at all: there is no individual in class i who chooses x. Then we say that people in class i are not free to choose x.

Suppose that the distribution in all classes were the same. Then in all classes people would have "the same" liberty: in this case we would have the maximal degree of independence freedom. Statistically, this case arises when the marginal distributions are independent; hence, a measure of freedom is given by a measure of independence: the higher the independence, the lower the effect of characteristics on the available options, the higher the independence freedom.

Following this line of argument G-G propose to measure the independence freedom by measuring the statistical association for bivariate distributions. Hence they propose the following coefficient of dependence:

$$\varepsilon = -\sum_i \sum_j \frac{p_{ij}}{\log p_i} \log \frac{p_{ij}}{p_i p_j}$$

G-G propose then to combine the two measures of diversity and independence to obtain a measure of overall social freedom. Their freedom coefficient is:

$$f = D\,(1 - \varepsilon)$$

which appears as the product of the diversity factor D and the independence factor $(1 - \varepsilon)$. However, while this is pragmatically useful, there is little theoretical justification for the construction of such an index, and moreover we feel that perhaps the two aspects should be kept separate.

On the other hand, the formal approach underlined in section 3 can be employed to explore the link between choices and socially relevant and socially irrelevant characteristics, on the basis of the conception of justice prevailing in a given society. Thus, if we partition Y into 'socially relevant characteristics' U and 'socially irrelevant characteristics' V, concepts of statistical dependence for the trivariate distribution of X, U, V (see e.g. Dardanoni and Forcina (1998) and Bartolucci, Forcina and Dardanoni (2001)), can be fruitfully employed to explore the link between choices and individual characteristics.

4.3 An Example: the choice of profession

Let us use an example to illustrate G-G approach to measuring the diversity and independence aspects of freedom. We consider G-G example of the choice of a profession, contained in section 4, Table 1.

In our notation, the example can be written in the following terms: X has 4 elements (choices of occupation) and Y (set of individual characteristics) has 16 elements. The elements of Y are given by the Cartesian product of fathers' occupation V (socially irrelevant, with 4 elements) and ability (socially relevant, with 4 elements) U. This can be arrayed by 4 2×2 contingency tables, one for each ability group.

4.4 Freedom as diversity

The marginal distribution of X (that is, the overall distribution of people in the four professions) cna be analyzed (and compared with the distribution of professions in another society) by means of a diversity index as Shannon's entropy, as argued in section X. However, this leaves open the problems discussed above, for example how do you distinguish between a planned economy, where you have the same variety of jobs, but all imposed by the central

authority to citizens, and a free market economy? A possible solution, as hinted in section X, is to compare the distribution of profession which would emerge by the free choice of the citizens with the actual marginal distribution. However, the in the choice of the profession we clearly have two sides of the market which have to be matched: in other words, the ''desires" of the citizens have to be matched with the availability of the different professions, which clearly depends on the changing economic and social conditions of the society. This ''matching" aspect of the choice of profession is better captured be the concept of freedom as indpendence.

4.5 Freedom as independence

In this example a society is free when, conditionally on ability U, X is independent on V. Thus, on the one hand the level of discussion is then shifted to another plane, that of the appropriate choice of the what should be deemed as releveant and irrelevant characteristics (see Pettit, 2001 for a defence of the concept of ''freedom as fitness to be held responsible"); on the other hand, once we have agreed on the appropriate choice of variables X, U, V, statistical techniques are available for a rigorous and objective comparison of overall freedom.

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