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UNCOVERED BARGAINING SOLUTIONS

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Uncovered Bargaining Solutions

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Abstract. An *uncovered bargaining solution* is a bargaining solution for which there exists a complete and strict relation (tournament) such that, for each feasible set, the bargaining solution set coincides with the uncovered set of the tournament. We provide a characterization of a class of uncovered bargaining solutions.

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1. Introduction

A bargaining solution expresses 'reasonable' compromises on the division of a surplus within a group. In this paper we ask the following question: given a bargaining solution, does there exist a complete and strict relation T (a tournament) such that, for each feasible set A, the bargaining solution set coincides with the uncovered set of T restricted to A? If the answer is positive, we call the bargaining solution an *uncovered bargaining solution*.

We offer two (related) motivations. First, a bargaining solution can be interpreted as a fair arbitration scheme (as argued for instance in Mariotti [9]). In this sense, we may think of a bargaining solution as being ratified (or ratifiable) by a *committee*. In this interpretation, the tournament expresses the majority preferences of the committee, and the uncovered set is the solution to the majority aggregation problem. A bargaining solution that does not coincide with the solution of any tournament is certainly not fair in the described sense: it could not be ratified by any committee.

A second interpretation follows the 'group revealed preference' interpretation pioneered by Peters and Wakker [11]. As they argue, 'the agreements reached in bargaining games may be thought to *reveal* the preferences of the bargainers as a group' (p. 1787). A tournament is a non-standard type of preference (lacking transitivity), which has recently been considered in individual choice theory (Ehlers and Sprumont [4], Lombardi [7]). It seems even more appropriate to consider such non-standard preference for a group than for an individual.

For *single valued* solutions the issue under study has essentially been solved, since a single valued uncovered bargaining solution *maximizes* (if certain regularity conditions are met)[1] a binary relation (in other words, the solution point is a Condorcet winner of the underlying tournament). For the domain of convex problems, Peters and Wakker [11] have shown that this is the case if and only if the solution satisfies Nash's Independence of Irrelevant Alternatives[2]. Denicolò and Mariotti [3] show that

[1] See the end of the next section for a discussion of this point.

[2] Peters and Wakker work with a weak relation. However it is easy to show - by using elementary duality properties in the maximization of binary relations - that a strict relation could be used instead. See e.g. Kim and Richter [6] or Aleskerov and Monjardet [1] for discussions of this issue in

the same holds for certain domains of non-convex problems, provided that Strong Pareto Optimality is assumed. In this latter case the binary relation is transitive. Therefore, the problem under study is new and interesting only for multivalued solutions. It is thus natural to look at a domain of nonconvex problems, as many notable solutions (such as the Nash Bargaining Solution) are single-valued on a domain of convex problems.

We focus on solutions which satisfy a 'resoluteness' condition: loosely speaking, when only two feasible alternatives x and y are Pareto optimal (so the bargaining problem is essentially binary), the solution picks either x or y. For this class of solutions, we provide a complete characterization of uncovered bargaining solutions for which the underlying tournament satisfies certain Paretian properties. The characterization uses four axioms: Strong Pareto Optimality; a standard Expansion property (if an alternative is in the solution set of a collection of problems, it is in the solution set of their union); a generalization of the 'Condorcet' property (if an alternative is chosen in 'binary' comparisons over each alternative in a collection, then it is the solution of the problem including all the alternatives in the collection); and a weak contraction consistency property (implied by Arrow's choice independence axiom).

2. Preliminaries

An n-person bargaining problem is a pair (A, d), with $d \in A$ and $A \subseteq \Re^n$, where A represents the set of feasible *alternatives* and d is the *disagreement point.*

The null-vector is denoted $\mathbf{0} \in \Re^n$. The vector inequalities in $\Re^n$ are: $x > y$ (resp.: $x \geqslant y$) if and only if $x_i > y_i$ (resp.: $x_i \geqslant y_i$) for every i. We view, as usual, $x \in \Re^n$ as a utility or welfare vector for n agents.

A domain of bargaining problem $\mathcal{B}$ is said to be *admissible* if:

D1 For every pair $(A, d) \in \mathcal{B}$: A is compact, and there exists $x \in A$ such that $x > d$.

D2 For all $x, y \in \Re^n_d$, where $x \neq y$ and $\Re^n_d = \{x \in \Re^n | x > d\}$, there exists a unique $(M(x, y), d) \in \mathcal{B}$ such that:

abstract choice theory.

1) $x, y \in M(x, y)$ and for every $z \in M(x, y)$ such that $z \notin \{x, y\}$, $x \geqslant z$ or $y \geqslant z$;

2) for every $(A, d) \in \mathcal{B}$ such that $x, y \in A$: $M(x, y) \subseteq A$.

D3 For all $(A, d), (B, d) \in \mathcal{B}$: $(A \cup B, d) \in \mathcal{B}$.

Many bargaining domains considered in the literature are particular cases of admissible domains[3]. For example the set of comprehensive problems (Zhou [12], Peters and Vermeulen [10]), the set of finite problems (Mariotti [8], Peters and Vermeulen [10]), the set of all problems satisfying D1 (Kaneko [5]), the set of d-star shaped problems[4]. D2 guarantees the existence of a 'minimal' problem containing any two given alternatives x and y, and such that x and y are the only strongly Pareto optimal feasible alternatives.

Unless specified otherwise, $\mathcal{B}$ is from now on a class of n-person admissible bargaining problems. A bargaining *solution* on $\mathcal{B}$ is a nonempty correspondence $f : \mathcal{B} \rightrightarrows \Re^n$ such that $f(A, d) \subseteq A$ for all $(A, d) \in \mathcal{B}$.

Given a bargaining solution f, we say that an alternative $x \in A$ is the f-*Condorcet winner* in $(A, d) \in \mathcal{B}$, denoted by $x = CW(A, d)$, if $x = f(M(x, y), d)$ for all $y \in A$, with $y \neq x$. Moreover, $x \in A$ is said to be an f-*Condorcet loser* in (A, d), denoted by $x \in CL(A, d)$, if $y = f(M(x, y), d)$ for all $y \in A$, with $y \neq x$.

Finally, the following abuses of notation will be repeated throughout this note: $f(A, d) = x$ instead of $f(A, d) = \{x\}$, $A \cup x$ instead of $A \cup \{x\}$, $A \backslash x$ instead of $A \backslash \{x\}$.

We consider only *resolute* solutions, that is those which satisfy the following property. For all $x, y \in \Re^n$, with $x \neq y$, for all $A \subseteq \Re^n$:

Resoluteness: $|f(M(x, y))| = 1$.

Resoluteness is analogous to a property with the same name imposed by Ehlers and Sprumont [4] and Lombardi [7] for individual choice functions over finite choice sets, given that (in the presence of Strong Pareto Optimality, defined below) the

[3] This class was essentially introduced in Denicolò and Mariotti [3].

[4] That is, those problems (A, d) for which the convex hull of $\{d, x\}$ is in A for all $x \in A$.

minimal problem $M(x,y)$ involves essentially a choice between only two alternatives. For standard solutions that are obtained by maximizing a quasiconcave 'social welfare function' (e.g. the Nash Bargaining Solution or the Utilitarian solution) this involves adding a tie-breaking criterion on minimal problems.

In addition the following properties will be used in the characterization result.

Axiom 1. ***(Strong Pareto Optimality)*** *$x \geqslant y$ and $x \neq y \in f(A,d) \Rightarrow x \notin A$.*

Axiom 2. *$A, B \in \mathcal{B}$, $x = CW(A,d)$ & $y \in CL(B \cup x, d) \Rightarrow y \notin f(A \cup B, d)$*

Axiom 3. *$x, y, z \in \Re^n$, with $x \neq y \neq z$, $x = f(M(x,y), d)$ & $y = f(M(y,z), d)$ $\Rightarrow x \in f(M(x,y) \cup M(y,z), d)$*

Axiom 4. *Given a class of problems $\{A_k, d\}$, then $\cap_k f(A_k, d) \subseteq f(\cup_k A_k, d)$*

Strong Pareto Optimality is standard. Axioms 2 is a generalization of the natural 'Condorcet Winner Principle'

$$x = CW(A,d) \Rightarrow x = f(A,d)$$

which is implied by setting $B = \emptyset$ in axiom 2.

Axiom 3 is a weak independence property. It says that if an alternative x is the unique solution point in a minimal problem where the only other Pareto optimal feasible alternative is y, and if y is the unique solution point in a minimal problem where the only other Pareto optimal feasible alternative is z, then x is a solution point of a minimal problem where the only other Pareto optimal feasible alternatives are y and z. Consider the following standard contraction consistency axiom[5]: $R \subset S$ & $f(S,d) \cap R \neq \varnothing \Rightarrow f(R,d) = f(S,d) \cap R$. Suppose $x \notin f(M(x,y) \cup M(y,z), d)$. If f is Pareto optimal then $f(M(x,y) \cup M(y,z), d) \subseteq \{y, z\}$. Suppose $y \in f(M(x,y) \cup M(y,z), d)$. If contraction consistency holds, then $f(M(x,y), d) = y$. If on the other hand $y \notin f(M(x,y) \cup M(y,z), d)$, that is $z = f(M(x,y) \cup M(y,z), d)$, and if f satisfies contraction consistency, then

[5] This is also called Arrow's choice independence axiom.

$z = f(M(y,z), d)$. In either case the premise of axiom 3 is violated. This shows that, in the presence of Pareto optimality, axiom 3 is a very special implication of contraction consistency.

Finally Axiom 4 is standard in choice theory: if an alternative is a solution point for every element of a given collection of bargaining problems, then it is still a solution point of their union.

We are, as usual, only interested in solutions that satisfy translation invariance. Then, we can set $d \equiv \mathbf{0}$. A bargaining problem simply becomes a subset of $\Re^n$ containing the null-vector and the notation is simplified accordingly.

A binary relation $T \subseteq \Re^n \times \Re^n$ is a *tournament* if it is asymmetric (i.e., for every $x, y \in \Re^n$, $x \neq y$, $(x,y) \in T \Rightarrow (y,x) \notin T$) and weakly connected (i.e., for every $x, y \in \Re^n$ with $x \neq y$, $\{(x,y), (y,x)\} \cap T \neq \emptyset$). We denote by $\mathcal{T}$ the set of all tournaments on $\Re^n$. A restriction of T to $A \subseteq \Re^n$, denoted by $T|A$, is a tournament.

For $x \in \Re^n$, let $T^{-1}(x)$ and $T(x)$ denote the *lower* and *upper sections* of T at x, respectively, that is:

$$\begin{aligned} T^{-1}(x) &= \{y \in \Re^n |\, (x,y) \in T\}, \text{ and} \\ T(x) &= \{y \in \Re^n |\, (y,x) \in T\}. \end{aligned}$$

For any tournament $T \in \mathcal{T}$ and $A \subseteq \Re^n$, define its covering relation $C|A$ on A by:

$$(x,y) \in C|A \text{ iff } (x,y) \in T|A \text{ and } T^{-1}(y) \cap A \subset T^{-1}(x) \cap A$$

The uncovered set of $T|A$, denoted $UC(T|A)$, consists of the $C|A-$maximal elements of A, that is:

$$UC(T|A) = \{x \in A |\, (y,x) \notin C|A \text{ for all } y \in A\}.$$

The *Strong Pareto* relation P on $\Re^n$ is defined by

$$\text{for } x, y \in \Re^n,\ x \neq y\text{: } (x,y) \in P \Leftrightarrow x_i \geq y_i \text{ for all } i, \text{ and } x_j > y_j \text{ for some } j.$$

We say that a tournament $T \in \mathcal{T}$ is *Pareto consistent* if for $x, y, z \in \Re^n$, with $x \neq y \neq z$:

$$(x,y) \in P \Rightarrow (x,y) \in T,$$

$$(x, y) \in P \ \& \ (y, z) \in T \Rightarrow (x, z) \in T.$$

So, a Pareto consistent tournament includes the Strong Pareto relation and satisfies a form of 'Pareto transitivity': any x which Pareto dominates y will beat any alternative z which is beaten by y.

Definition 5. *A bargaining solution f is an uncovered set bargaining solution (UCBS) if there exists $T \in \mathcal{T}$ such that, for every $A \in \mathcal{B}$, $f(A) = UC(T|A)$. In this case we say that T rationalizes f.*

As an example of an UCBS which does not coincide with a standard solution, consider the following class. Let F be a asymmetric transitive and weakly connected relation, which here we interpret as 'fairness'[6]. Recall that P is the Strong Pareto relation. Then define the solution f by: $x \in f(A)$ iff for all $y \in A \backslash x$: either $(x, y) \in P$; or $[(y, x) \notin P \ \& \ (x, y) \in F]$; or $[(x, z) \in P \ \& \ (z, y) \in F \ \& \ (y, z) \notin P$ for some $z \in A]$; or $[(x, z) \in F \ \& \ (z, x) \notin P \ \& \ (z, y) \in P$ for some $z \in A]$. In words, fairness is ignored if and only if a Pareto ranking is possible, and given this constraint, for any other alternative y, the chosen alternative x must either dominate y directly in terms of Pareto or fairness, or indirectly via an intermediate alternative z, applying the Pareto and fairness (or vice versa) criteria in succession. The solution f is, in each problem, the uncovered set of the tournament T defined by: $(x, y) \in T$ iff either $(x, y) \in P$; or $[(y, x) \notin P$ and $(x, y) \in F]$ (note that T is weakly connected and asymmetric); or both.

Finally, we come back briefly to the issue of single-valued solutions alluded to in the introduction. Let T be a tournament on A, and suppose $UC(T|A)) = \{x\}$ for some $x \in A$. If x is not a Condorcet winner, $T(x)$ is nonempty. Let $y \in UC(T|T(x) \cup x)$. Then $y \in UC(T|A)$, since for any $z \in T^{-1}(x)$ we have $(y, x), (x, z) \in T$. But this contradicts the assumption that $UC(T|A) = \{x\}$. So x must be a Condorcet winner of A if it is the unique uncovered element of A. In this reasoning, however, it assumed that the uncovered set of $T(x) \cup x$ is nonempty,

[6] F could be constructed for example on the basis of the Euclidean distance to the 45^0 line, with the addition of a tie-breaking criterion.

which is not necessarily true if $T(x)$ is not finite. For conditions guaranteeing the nonemptiness of the uncovered set on general topological spaces see Banks, Duggan and Le Breton [2].

3. Characterization

We show below that in the presence of Resoluteness, axioms 1-4 characterize uncovered bargaining solutions for which the rationalizing tournament is Pareto consistent.

Theorem 6. *Let f be a resolute bargaining solution. Then f is an UCBS, rationalized by a Pareto consistent tournament, if, and only if, it satisfies axioms 1-4.*

Proof. (*Only if*). Let f be a resolute UCBS. Obviously f satisfies Strong Pareto Optimality and Weak Expansion. Next, we check axioms 2-3.

To verify axiom 2, let $x = CW(A)$, and $y \in CL(B \cup x)$, with $x \neq y$. The existence of a Pareto consistent T implies that $(x,z) \in T$ for all $z \in A \backslash x \cup y$. Moreover, as $y \notin f(M(y,w))$ for all $w \in B \backslash y$, there exists $w' \in M(w,y) \backslash y$ which covers y. If $w' = w$, then $(w,y) \in T$. Otherwise, consider $w' \neq w$. Since w' is not strongly Pareto dominated by y, it must be the case that $(w,w') \in P$, by D2. It follows from Pareto consistency of T that $(w,y) \in T$. Therefore, whether or not $w = w'$ we have that $(w,y) \in T$. Since $(x,z) \in T$ for all $z \in A \backslash x \cup y$ and $(w,y) \in T$ for all $w \in B \backslash y$, it follows that x covers y, and so $y \notin UC(T|A \cup B)$ as desired.

For axiom 3, let $x, y, z \in \Re^n$, with $x \neq y \neq z$, and let $x = f(M(x,y))$ and $y = f(M(y,z))$. We show that $x \in f(M(x,y) \cup M(y,z))$. Since $x = f(M(x,y))$ and $y = f(M(y,z))$, there exists a Pareto consistent T such that $(x,x') \in T$ for all $x' \in M(x,y) \backslash x$ and $(y,y') \in T$ for all $y' \in M(y,z) \backslash y$. Observe $M(x,y) \cup M(y,z) \in \mathcal{B}$, by D3. Since no point in $M(x,y) \cup M(y,z) \backslash x$ covers x, it follows that $x \in f(M(x,y) \cup M(y,z))$.

(If). Let f be a resolute bargaining solution satisfying the axioms. Define the relation T on $\Re^n$ as follows:

$$\text{for all } x, y \in \Re^n, \text{ with } x \neq y,\ (x,z) \in T \text{ iff } x = f(M(x,y)).$$

For all $x, y \in \Re^n$, with $x \neq y$, there exists a minimal problem $M(x,y)$, by D2. It follows from Strong Pareto Optimality and Resoluteness that either $x = f(M(x,y))$

or $y = f(M(x,y))$. Then, T is weakly connected and asymmetric, and so $T \in \mathcal{T}$. To see that T is Pareto consistent as well, let $x, y, z \in \Re^n$, with $x \neq y \neq z$. We show that $i)$ $xPy \Rightarrow xTy$, and $ii)$ $(x,y) \in P$ & $(y,z) \in T \Rightarrow (x,z) \in T$. Case $i)$ directly follows from Strong Pareto Optimality. Next, we show case $ii)$. Since $x = f(M(x,y))$ and $y = f(M(y,z))$, it follows from axiom 3 combined with D3 that $x \in f(M(x,y) \cup M(y,z))$. Since $M(x,y) \cup M(y,z) = M(x,z)$, Resoluteness implies that $x = f(M(x,z))$, and we are done.

We claim that

$$f(A) = UC(T|A) \text{ for all } A \in \mathcal{B}.$$

Fix $A \in \mathcal{B}$. For any $x \in A$ partition A in $T(x)$, $T^{-1}(x)$ and $\{x\}$.

Let $x \in f(A)$ and assume, to the contrary, that x is a covered point. Then for some $y \in A \backslash x$ it must be the case that $(y,x) \in T$ and $T^{-1}(x) \subset T^{-1}(y)$. Therefore $y = CW(T^{-1}(x) \cup \{x,y\})$. Let $z \in T(x)$, and consider the minimal bargaining problem $M(x,z)$. By definition of T, we have that $z = f(M(x,z))$ for all $z \in T(x)$, and so $x \in CL(T(x) \cup x)$. It follows from axiom 2 that $x \notin f(A)$, a contradiction.

Conversely, let $x \in UC(T|A)$. Take any $y \in T^{-1}(x)$, and consider the minimal bargaining problem $M(x,y)$. By definition of T it follows that $x = f(M(x,y))$. Because it is true for any $y \in T^{-1}(x)$, we have that $x = CW(T^{-1}(x) \cup x)$. If $T(x) = \varnothing$, it follows from the Condorcet Winner Principle implied by axiom 2 that $x \in f(A)$. Otherwise, take any $z \in T(x)$. Since T is Pareto consistent and $z \in T(x)$, there exists $y \in T^{-1}(x)$ which is not strongly Pareto dominated either by x nor by z such that $(y,z) \in T$. Axiom 3, combined with D3, implies that $x \in f(M(x,y) \cup M(y,z))$. Because this holds for any $z \in T(x)$, axiom 4 implies that $x \in f(A)$.

4. Independence of the axioms

The axioms used in theorem 6 are tight, as argued next.

For an example violating only Strong Pareto Optimality, consider the disagreement point d as the solution of any admissible bargaining problem, that is, $f(A,d) = d$ for every $(A,d) \in \mathcal{B}$. Clearly, f is resolute and satisfies axioms 2-4, but not Strong Pareto Optimality.

Next, let us consider for simplicity only 2-person bargaining problems.

For an example violating only axiom 2, define, for every $x, y \in \Re^2_+$, with $x \neq y$:

$$f(M(x,y)) = x \text{ if } x_1 + x_2 > y_1 + y_2 \text{ or } x_1 + x_2 = y_1 + y_2 \;\&\; x_1 > y_1,$$

whilst, for any non-minimal problem $A \in \mathcal{B}$, define the bargaining solution f as:

$$f(A) = \arg\max_{s \in A}(s_1 + s_2).$$

To see that axiom 2 is contradicted, consider the domain of finite problems, and let $x, y, z \in A$, where $x = (2,1)$, $y = (1,2)$, and $z = (1,0)$. By definition, $f(xy) = f(xz) = x$, and $f(yz) = y$, but $f(xyz) = xy$, which violates axiom 2. Obviously, the bargaining solution is resolute, and it satisfies axioms 1 and 3-4.

For an example violating only axiom 3, fix $y, z \in \Re^2_{++}$, with $y \neq z$, such that $y_1 + y_2 = z_1 + z_2$. Define

$$f(M(z,y)) = z \text{ if } a = z \;\&\; b = y.$$

Given any other bargaining problem $A \in \mathcal{B}$, define the bargaining solution f as the following:

$$f(A) = \begin{cases} \arg\max_{s_1}\{\arg\max_{s\in A}(s_1+s_2)\} & \text{if } \; y \notin A \text{ or } z \notin A \\ \arg\max_{s_1}\{\arg\max_{s\in A}(s_1+s_2) - \{y\}\} & \text{otherwise} \end{cases}.$$

To see that axiom 3 is contradicted, consider the domain of finite problems, and let $x, y, z \in A$, where $x = (2,2)$, $y = (3,1)$, and $z = (1,3)$. We have that $f(xy) = y$, $f(xz) = x$, and $f(yz) = z$. Consider the bargaining problem $A' = \{x, y, z\}$. Given that $y, z \in A'$, it follows from definition of f that $x = f(A')$, which violates axiom 3. Clearly, the bargaining solution is resolute and satisfies axiom 1. It is easy but tedious to check that it satisfies axioms 2 and 4 as well (details available from the authors).

Finally, for an example violating only axiom 4, fix $x, y, z \in \Re^2_{++}$, with $x \neq y \neq z$ and $x_1+x_2 = y_1+y_2 = z_1+z_2$, and let $M(x,y) \cup M(y,z) = C \in \mathcal{B}$ with $f(M(x,y)) = x$, $f(M(y,z)) = y$, and $f(M(x,z)) = z$. Define for any $a, b \in \Re^2_+ \backslash \{x, y, z\}$, with

$a \neq b$: $f(M(a,b)) = a$ if $a_1 + a_2 > b_1 + b_2$ or $a_1 + a_2 = b_1 + b_2$ & $a_1 > b_1$, whilst let for any $a \in \Re^2_+ \backslash \{x,y,z\}$ and $b \in \{x,y,z\}$:

$$\begin{array}{lll} f(M(a,b)) = a & \text{if} & a_1 + a_2 > b_1 + b_2 \\ f(M(a,b)) = b & \text{if} & a_1 + a_2 \leq b_1 + b_2 \end{array}.$$

Define the following set of alternatives S_a:

$$S_a = \{b \in \Re^2_+ \backslash a | f(M(a,b)) = a\}$$

and for any bargaining problem $A \in \mathcal{B}$ not yet considered define the bargaining solution f as:

$$f(A) = \begin{cases} \arg\max_{s_1}(\arg\max_{s \in A}(s_1 + s_2)) & \text{if} \quad A \cap \{x,y,z\} = \emptyset \\ \arg\max_{s_1}(\arg\max_{s \in A}(s_1 + s_2) - S_a) & \text{if} \quad A \cap \{x,y,z\} = \{a\} \\ \arg\max_{s_1}(\arg\max_{s \in A}(s_1 + s_2) - S_a) & \text{if} \quad A \cap \{x,y,z\} = \{a,b\} \ \& \ f(M(a,b)) = a \\ x, y, z & \text{if} \quad A = C \\ \arg\max_{s_1}(\arg\max_{s \in A}(s_1 + s_2) - S_y) & \text{otherwise} \end{cases}.$$

To see that axiom 4 is contradicted, consider the domain of finite problems, and let $A = \{x,y,z,w\}$, where $x = (2,2)$, $y = (3,1)$, $z = (1,3)$, and $w = (1,1)$. By construction $f(xy) = x$, $f(yz) = y$, $f(xz) = z$, and $f(xyz) = xyz$; furthermore, we have that $f(xw) = x$, $f(yw) = y$, and $f(zw) = z$. Let us consider the bargaining problem $\{x,z,w\} = B$. Since $x, z \in B$ and $f(xz) = z$, it follows from the definition of f that $z = f(B)$. However, we have that $z \notin f(A)$, by definition of f, which violates axiom 4. The bargaining solution as defined above is obviously resolute and it satisfies 1. Moroeover, it can be checked that it satisfies axioms 2-3 (the tedious analysis is available from the authors).

5. CONCLUDING REMARKS

Lombardi [7] studies choice correspondences on the domain of all subsets of an abstract finite set, and poses the same question as this paper. At the technical level, the main difficulty here is that bargaining sets are not always finite. This necessitates the different axioms and argument of proof presented in this paper, as well as the restriction to Pareto consistent tournaments. These arguments exploit heavily the ordering structure of $\Re^n$ and the natural Strong Pareto Optimality assumption, which is instead meaningless on the domain considered by Lombardi.

Ehlers and Sprumont [4], on the same domain as Lombardi, characterize choice correspondences for which there exists a tournament such that, for each choice set, the choice is the *top cycle* of the tournament. It is natural to seek a similar characterization in the context of bargaining solutions, as we have done for the uncovered set. This remains an open question for future research.

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