XXI CONFERENZA
PUBLIC CHOICE E POLITICAL ECONOMY
I fondamenti positivi della teoria di finanza pubblica
Pavia. Università. 24-25 settembre 2009

A MODEL OF ASYMMETRIC FOMC DIRECTIVE

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March 19, 2009

PRELIMINARY AND UNCOMPLETE!

Several other members indicated that they would have preferred to tighten at that meeting. ... The asymmetric directive, which held prospect of near-term tightening, once again allowed FOMC to reach a consensus.

Meyer (2004), page 83

Abstract

We investigate policy outcomes in a dynamic infinite-horizon bargaining model under two bargaining protocols. First one, 'without the directive', captures the standard way monetary policy committees take decisions, most importantly the endogenous nature of the default policy. Second one, 'with the directive', is inspired by the decision protocol of Federal Open Market Committee, a decision body of the Federal Reserve. The key difference is that under this bargaining protocol chairman's offers are not restricted to those where today's inflation decision is the default policy during the next committee meeting.

We provide existence and uniqueness results for both versions of the model, explicitly derive the equilibrium for the model without the directive and estimate the equilibrium for the model with the directive.

We show that without the directive policy-makers may fail to reach an agreement even when their current preferences are identical for fear of giving up their bargaining position which is valuable in the future disagreement periods.

On the other hand, we prove that in any equilibrium with the directive committee decisions during the periods when policy-makers have identical current preferences fully reflect their common will despite the possibility of future disagreements.

We take this as an evidence of the directive serving consensus building role during the FOMC decision making process, an idea discussed in the empirical literature on the topic.

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1 Introduction

This paper is an attempt to develop a bargaining model of central bank decision committee to shed more light on the resulting central bank behaviour.

Most of the existing models suffers from several shortcomings. The older literature on the dynamic inconsistency of low inflation monetary policy almost exclusively abstracts from the fact that the monetary policy in most cental banks is not set by a single individual but by a committee.

Focusing on the papers that explicitly model central bank led by a committee most of them lack any sort of strategic interaction between the committee members. Furthermore, most of the papers focus on a single period models and abstract from any dynamic interactions.

To address those shortcoming, this paper sets up a model in which central bank is led by a committee. In order to investigate the nature of strategic interactions the committee is populated by agents that do not always agree on the best course of the monetary policy. Setting up a model with explicit time dimension also allows us to capture the dynamic aspect of central bank decisions. The policy enacted today constitutes a default option during the next policy decision meeting.

The key research question is the impact of the arrangement in which the committee at a given meeting decides not only about the policy for current period, but also about the default policy for the next meeting. This differs from the usually practice in that it allows the current policy and the next meeting default policy to differ. Such an arrangement has been used for over two decades by Federal Open Market Committee (FOMC), monetary policy decision body of the Federal Reserve System.

At the end of each meeting, FOMC issued a directive specifying not only its decision about the target federal funds rate but also about its 'bias' for the future. The bias has been either *symmetric* or *asymmetric* towards tightening or easing. We build a model which incorporates this bargaining protocol and compare its equilibrium with the equilibrium of the model where policy implemented today is the default policy for the next round of a bargaining.

We provide existence and uniqueness results for both versions of the model and explicitly solve the bargaining model without the directive. For the model capturing the FOMC bargaining protocol we partially characterize the resulting equilibrium and provide numerical examples for cases where closed form solutions are hard to obtain.

In terms of results, we show that the bargaining protocol without the directive prevents policy-makers from reaching consensus even in the periods when their current preferences are the same and that the dynamic bargaining results in policy inertia. For the FOMC bargaining protocol with the directive, we show that it can play consensus building role.

We proceed as follows. Section 2 surveys the related literature and section 3 describes in detail the FOMC decision making procedure. Section 4 lays out the model which captures the distinction between FOMC and standard decision making. We solve the two versions of the model in section 5. The we conclude.

All the proofs are relegated to the appendix.

2 Survey of Literature

The paper is related to several strands of literature. On the most general level it belong into the strand of literature on dynamic inconsistency of low inflation monetary policy that started with Kydland and Prescott (1977). Although interesting we do not provide full survey of this literature (see Persson and Tabellini (2000) and Drazen (2000) for surveys).

The more relevant literature explicitly models central banks led by a committee rather than an individual. An early paper Cothren (1988) shows how low inflationary monetary policy can be sustained through reputation building central bank. His model avoids a peculiar feature of the low inflation through reputation result originally derived by Barro and Gordon (1983), namely perfectly coordinated trigger strategy used by inflation expectations forming agents.

Creation of European Central Bank has also generated interest in the committee based central bank models. Matsen and Roisland (2005) investigate an impact of different voting rules for inflation and output in a committee composed of different country representatives. At the same time Fatum (2006) derives low inflation result in a model with monetary policy committee. His result hinges on the fact that representatives of inflation-prone countries cannot propose negative interest rate. In a framework where the final policy is given by the weighted mean of individual proposals this gives an advantage to inflation-averse countries in preceding strategic delegation game.

Gerlach-Kristen (2006) offers a rationale for monetary policy to be decided by a committee in a model where committee members observe imperfect signals about unobservable output gap. In this way committee as a whole obtains better information about the unobservable that defines the optimal policy decision. She investigates the impact of using different decision rules on the quality of committee decisions.

Another low inflation through committee result is offered by Dal Bo (2006). It rest on the specific nature of the voting in his model. Final inflation is the result of the process which starts with zero inflation as a default and voting takes place over increments until (super)majority fails to support the new proposal. In this way inflation unravels only to the border of the (super)majority core which is lower than the inflation that would prevail in a model with a single policy maker.

With somewhat different focus Waller (1989, 1992, 2000) investigates models of partisan appointment to central bank committee that subsequently decides on the course of monetary policy via majority voting. Chang (2003) runs in a similar spirit.

Majority of the models mentioned above lack any sort of strategic interaction among the committee members. Resulting policy is often the median of the policies preferred by the individual committee members or their (weighted) mean. Complemented by the fact that the policies preferred by the individual members are assumed to be a common knowledge leaves no room for strategic interaction within a committee. Furthermore, most of the models lack explicit time dimension and hence are inherently static.

Mihov and Sibert (2006) and Sibert (2003) are among the models that focus on the strategic interaction. Both papers investigate the model with committee composed of two members both of whom are of either hawk or dove type which is assumed to be a private information. Low inflation result is derived through the desire of the inflation-prone dove to acquire reputation which makes subsequent inflationary surprises less costly.

Despite the strategic interaction within monetary policy committee both papers treat what happens in the case of committee members' disagreement as exogenous. While the former paper assumes that in the case of disagreement implemented policy is weighted mean of members' preferred ones the later assumes exogenously specified default policy.

Riboni (2009) avoids the need to specify default policy by using dynamic bargaining model. In his paper committee composed of fixed agenda setter (chairman) and ordinary members decides on the monetary policy. Each period chairman proposes an alternative that is then pitched against the status-quo in a majority voting, status-quo being the policy implemented in the previous period.

Low inflation monetary policy is credible in this model since, conditional on low inflation expectations, unprofitable deviations (inflation surprises) are not proposed by the chairman and profitable deviations are not accepted by the committee.

Despite the possibility of achieving low inflationary monetary policy in a model with monetary policy committee another problem arises in that committee can produce considerable policy inertia, point stressed by Blinder (1998) (see experimental results in Blinder and Morgan (2000) that point to the contrary).

This is exactly the point illustrated by Riboni and Ruge-Murcia (2008) within a similar framework as Riboni (2009). In the model where committee members' preferences diverge in certain periods even in the agreement periods common-optimal policy might not be implemented. This is due to the dynamic bargaining framework in which ordinary member knows that now-optimal policy will put him in unfavourable bargaining position in the next period should the committee members disagree.

On the empirical side several papers deal with various aspects of asymmetric FOMC directive. Obvious question is whether the bias in the directive signals future moves in the monetary policy. In this respect Pakko (2005) provides evidence showing that the bias in the directive has a predictive power regarding future monetary decisions. On the other hand, Thornton and Wheelock (2000) and for the pre-1999 also Ehrmann and Fratzscher (2007) speak to the contrary.

Another possibility is that bias in the directive is used by the FOMC as a means of approving chairman's inter-meeting changes of the interest rate. This role of the asymmetric directive is confirmed by Lapp and Pearce (2000) but rejected by Thornton and Wheelock (2000) and Chappell, McGregor, and Vermilyea (2007).

Last possibility is that separating current policy from the future status-quo serves a consensus building purpose during the committee bargaining. This view is supported by the evidence in Ehrmann and Fratzscher (2007), Meade (2005) and Thornton and Wheelock (2000) but Chappell et al. (2007) provide evidence favouring the opposite. Overall, the empirical literature regarding the purpose of the asymmetric FOMC directive does not provide support for a single purpose explanation.

Lastly, Maier (2007) and Sibert (2006) survey economic and social psychological literature focusing on the implications for monetary policy committee design. In this respect the economic literature on Condorcet Jury theorem supports larger committees due to informational advantage in the imperfect information environments. On the other hand social psychological literature on group task effectiveness and group decision supports smaller committees that prevent free-riding of committee members and foster swift decision-making.

3 Institutional Background

The FOMC, monetary policy decision body of the Federal Reserve System, meets in every about six weeks. It comprises 7 members of Board of Governors of the Federal Reserve System and 12 presidents of Federal Reserve Banks. Members with voting power are all governors, president of the Federal Reserve Bank of New York and on a rotating basis four presidents of the Federal Reserve Banks.

Structure of the FOMC meetings at least for the Chairman Greenspan years is to start with staff report on economic conditions followed by 'economic go-round' and subsequently by 'policy go-round' (see Chappell et al. (2007) for more detailed description of FOMC meetings). In an economic go-round FOMC members took turns in explaining their view on the development of the economy.

Subsequent policy go-round usually started with Chairman's proposal that provided the reference for other speakers. At the end of the round Chairman proposed final policy including the target federal funds rate and the setting for the bias in the directive after which the formal voting took place in terms of 'assent' or 'dissent' statements.[1]

Asymmetric FOMC policy directive has been issued in its original form since 1983 until December 1999.[2] Apart from specifying current policy decision it also included a 'bias' which was either asymmetric towards tightening or easing or symmetric. For example directive biased towards tightening would say:

> In the context of the Committee's long-run objectives for price stability and sustainable economic growth, and giving careful consideration to economic, financial, and monetary developments, somewhat

[1] The description in Chappell et al. (2007) is derived from the transcripts of FOMC meetings published with five year lag which is why there is uncertainty regarding the current structure of FOMC meetings.

[2] See Thornton and Wheelock (2000) for detailed history and empirical evidence suggesting consensus building hypothesis regarding the role of the asymmetric directive.

> greater reserve restraint **would** or slightly lesser reserve restraint **might** be acceptable in the intermeeting period.
>
> *FOMC minutes from August 20, 1996 meeting*

Asymmetry towards tightening is exemplified by the use of word 'would' as opposed to 'might' in relation to restraint on commercial bank reserve positions. Symmetric directive would say:

> In the context of the Committee's long-run objectives for price stability and sustainable economic growth, and giving careful consideration to economic, financial, and monetary developments, slightly greater reserve restraint or slightly lesser reserve restraint **would** be acceptable in the intermeeting period.
>
> *FOMC minutes from January 30-31, 1996 meeting*

Since February 2000 asymmetry in the directive has been specified in terms of balance of risks assessment. Original wording has been to specify risks either for 'heightened inflation' or for 'economic weakness'. From May 2003 the balance of risks assessment includes the FOMC's view on both inflation and economic growth.

Regarding the timing of release of the asymmetry in the directive, until March 1999 it has been included in the minutes of FOMC meetings that has been published right after the next FOMC meeting. Since May 1999 it is included in the press release made public immediately after all the meetings.[3]

But the Federal Reserve is not the only central bank with similar provision. Recently Riksbank, Swedish central banks, has been explicitly referring to the future course of monetary policy. Press release issued after each monetary policy committee might read:

> Continued strong economic activity and rising inflation mean that the repo rate needs to be increased. It is reasonable to assume that the interest rate will need to be increased further, roughly in line with recent market expectations.
>
> *press release after December 15, 2006 meeting*

The press release might even refer to numerical value regarding the future interest rate. For example:

> It is also probable that the interest rate will need to be raised slightly further in the future. During the first half of 2008 the repo rate is expected to be around 4.25 per cent.
>
> *press release after October 30, 2007 meeting*

[3]On rare occasions FOMC opted not to specify its leaning. For example press release after the March 18, 2003 meeting reads '$\cdots$ Committee does not believe it can usefully characterize the current balance of risks $\cdots$'.

4 Model

To investigate how the possibility of having asymmetric directive influences conduct of monetary policy we investigate the simple model below. Our main question is whether having asymmetric directive results in different inflation outcomes and whether the asymmetric directive can be used as a consensus building mechanism.

The central bank in our model is governed by monetary policy committee composed of two members. The first member is a fixed chairman who has the policy proposal power and whom we denote by C (she). The second committee member is denoted by P (he) and has policy approval power. In other words the decision within the committee is done via majority voting between C's proposal and status-quo with ties decided in favour of the latter.

The utility of both policy-makers is given by

$$U_i = \sum_{t=0}^{\infty} \delta^t u_{i,t}$$

for $i \in \{C, P\}$ where δ is the common discount factor and $u_{i,t}$ is the per-period utility which is given by

$$u_{i,t} = -(p_t - \pi^* - \varepsilon_{i,t})^2 \tag{1}$$

where π^* is central banks's target inflation and $\varepsilon_{i,t}$ is random time-varying, i-specific preference shock.

Decisions about the monetary policy are done in the following way. At time t the committee convenes to make a decision. Up to their meeting policy in effect was p_{t-1} and they are convening knowing that the default policy for their meeting is q_{t-1}. At the meeting C proposes pair $\gamma_t = \{p_t, q_t\}$. If P agrees with the proposal γ_t is implemented and if he disagrees $\bar{\gamma}_t = \{p_t = q_{t-1}, q_t = q_{t-1}\}$ is implemented instead.

To prevent any confusion, in the text we refer to the $\{p_t, q_t\}$ pair as to a *policy*, call the policy with which the bargaining starts at time t, i.e. $\{q_{t-1}, q_{t-1}\}$, the *default policy* and reserve terms *inflation* and *status-quo* to the first and second elements of any policy.

To investigate the difference in the conduct of monetary policy with and without the asymmetric policy directive, we contrast two versions of the model. First, without the asymmetric directive, restricts C's proposals to those where current inflation and the next period status-quo are equal, in other word to the proposals that satisfy $\gamma_t = \{p_t = q_t, q_t\}$. This is the way most central bank committees operate. The interest rate in the current period is the default one by the time of a next meeting.

The second version, with asymmetric policy directive, tries to capture the FOMC arangement in that C's proposals are not restricted as above. For discussion convenience, we call the two versions of the model with and without directive respectively, omitting the 'asymmetric' adjective and when we refer to bargaining protocol, we have in mind this feature that distinguishes the two versions.

The timing of actions in period t is as follows. First, nature determines $\varepsilon_{i,t}$. Second, committee convenes with $\varepsilon_{i,t}$ being common knowledge. Third, C proposes γ_t against $\bar{\gamma}_t$ and P either agrees or disagrees after which winning option is implemented and the bargaining moves into period $t+1$.

To close the model, we assume specific distribution of the preference shocks. The assumption is that in certain periods C and P agree in which case their preference shocks are equal to zero. In the disagreement periods that happen with probability p_d the committee does not agree on the best course of monetary policy (see Chappell, McGregor, and Vermilyea (2005) or Meade and Sheets (2005) for evidence of diverging preferences of FOMC members). We assume that preference shocks in the disagreement periods are equal to

$$\varepsilon_{i,t} = \begin{cases} \phi & \text{for} \quad i = P \\ -\phi & \text{for} \quad i = C \end{cases}$$

with $\phi > 0$ or in other words in the disagreement periods P prefers higher inflation compared to C. We denote the disagreement periods by D and agreement periods by A.

Finally, we assume that if C cannot offer any policy γ_t which gives her higher utility than the default policy $\bar{\gamma}_t$ she offers $\bar{\gamma}_t$. In the similar spirit, we assume that if P is indifferent between γ_t and $\bar{\gamma}_t$, he votes for the γ_t. With this assumption on the equilibrium path offered policies will always be accepted and hence implemented so in the discussion we do not need to distinguish between policies C offers and those that eventually become effective.

Two period model

To build an intuition for the results below, we first solve the two period version of the model. Observing that in the last period when $t = 1$ the bargaining protocol plays no role, it readily follows that $p_{A,1} = \pi^*$. In the D periods, the policy will in general depend on the default policy. It is easy to show that $t = 1$ period inflation in D periods as a function of $t = 0$ period status-quo is

$$p_{D,1}(x) = \begin{cases} x & \text{for} \quad x \in \langle \pi^* - \phi, \pi^* + \phi \rangle \\ 2(\pi^* + \phi) - x & \text{for} \quad x \in \langle \pi^* + \phi, \pi^* + 3\phi \rangle \\ \pi^* - \phi & \text{otherwise.} \end{cases}$$

The intuition is following. In the A periods both policy-makers have the same preferences and they readily agree on the policy they both prefer. As their bargaining position in the future is not influenced by today's policy, there is nothing to prevent them from reaching consensus.

On the other hand, in the D periods their preferences differ. If the default policy happens to fall into the interval between their most preferred policies there is no way they can agree on something else. This is the first case above. If the default policy happens to be higher than P's most preferred policy, C will offer policy that makes P indifferent between γ_1 and $\bar{\gamma}_1$ but as close as possible to her most preferred policy. Under quadratic utility this amounts to offering γ_1 with the same distance from $\pi^* + \phi$ as the $\bar{\gamma}_1$ but closer to C's optimum. This is the second case above. With the default policy still further from the P's most preferred point, he is willing to accept wide range of policies one of which is the C's most preferred policy. Note that in this region, the outcome of the bargaining does not depend on the default policy.

Plugging the equilibrium inflation into the utility functions and taking expectations given the information at $t = 0$, C's expected utility as a function of $t = 0$ policy is

$$\mathbb{E}\left[U_{C,0}(x)\right] = \begin{cases} -\,p_d(x - \pi^* + \phi)^2 & \text{for } \; x \in \langle \pi^* - \phi, \pi^* + \phi \rangle \\ -\,p_d(\pi^* + 3\phi - x)^2 & \text{for } \; x \in \langle \pi^* + \phi, \pi^* + 3\phi \rangle \\ \;\; 0 & \text{otherwise} \end{cases}$$

and P's expected utility is

$$\mathbb{E}\left[U_{P,0}(x)\right] = \begin{cases} -\,p_d(x - \pi^* - \phi)^2 & \text{for } \; x \in \langle \pi^* - \phi, \pi^* + 3\phi \rangle \\ -\,4\phi^2 p_d & \text{otherwise.} \end{cases}$$

Proceeding to the first period $t = 0$, the outcomes will differ depending on the type of the period, bargaining protocol and on the default policy $\bar{\gamma}_0$ which is inevitably exogenous.

Without the directive in the D periods, the equilibrium policy as a function of the default policy is

$$p_{D,0}(x) = \begin{cases} x & \text{for } \; x \in \langle \pi^* - \phi, \pi^* + \phi \rangle \\ 2(\pi^* + \phi) - x & \text{for } \; x \in \langle \pi^* + \phi, \pi^* + 3\phi \rangle \\ \pi^* - \phi & \text{otherwise.} \end{cases}$$

The intuition behind the result is rather simple. For intermediate values of $\bar{\gamma}_0$, as the preferences of the policy-makers differ with respect to inflation as well as with respect to the status-quo for the next period, there is no way they can reach consensus on something else than $\bar{\gamma}_0$. If the $\bar{\gamma}_0$ happens to be above P's most preferred policy $\pi^* + \phi$, C will offer policy 'on the other side' of P's acceptance set which is closer to C's optimum. Yet for higher values of the default policy, P is made better of by C (at least) bringing the policy to her most preferred one.

It is not hard to show that the equilibrium policy, that is both inflation and status-quo, in the model with the directive for the D periods is exactly the same as for the model without the directive. Maybe little surprisingly, C chooses not to increase inflation in an attempt to gain better bargaining position by lowering the status-quo (or vice versa). We will see below that this result is specific to the two-period version of the model with the directive and does not hold in general.

Proceeding to the A periods, under the bargaining protocol without the directive the equilibrium inflation is

$$p_{A,0}(x) = \begin{cases} x & \text{for } \; x \in \langle \pi^* - \phi\kappa, \pi^* + \phi\kappa \rangle \\ 2(\pi^* + \phi\kappa) - x & \text{for } \; x \in \langle \pi^* + \phi\kappa, \pi^* + 3\phi\kappa \rangle \\ \pi^* - \phi\kappa & \text{otherwise} \end{cases}$$

where $\kappa = \frac{\delta p_d}{1+\delta p_d}$. The intuition for the result is the same as above. Note however that A periods are those when both policy-makers have equal preferences. The reason why they fail to agree on π^* which is inflation they would both prefer for $t = 0$ is that by doing so they would have to compromise on their bargaining position for $t = 1$. And as the bargaining position next period is given by q_0 which is by assumption equal to p_0, the default policy prevails.

Another thing to note is the fact that the bargaining position matters in the $t = 1$ period only if it is a D period. It is easy to confirm κ increases with both, the probability of D periods and with the discount factor δ. Hence higher is the p_d or δ the larger is the interval over which the default policy determines the equilibrium one.

In contrast, under the bargaining protocol with the directive equilibrium inflation is always equal to π^*. The logic behind this result is that in the A periods policy-makers' preferences are aligned along the inflation dimension and as the inflation can differ from the status-quo for the next period, C does not compromise her bargaining position by offering π^*.

At the same time, by being offered π^*, P is made better of and C will use this extra room to maneuver to improve her bargaining position for the next period. Hence provided C cannot offer $\{\pi^*, \pi^* - \phi\}$ which is the best she can do, she will set q_0 so as to make P indifferent between $\bar{\gamma}_0$ and γ_0.

5 Infinite horizon model

This section solves the infinite horizon dynamic bargaining problem for the two bargaining protocols. We focus on Stationary Markov Perfect Equilibria (S-MPE) where strategies in a given period depend only on the type of the period and the default policy for that period, i.e. only on the payoff relevant variables.

For technical reasons we restrict the policy space along any dimension to lie in the convex compact subset X of $\mathbb{R}$. Hence $\gamma_t, \bar{\gamma}_t \in X^2 \subseteq \mathbb{R}^2$. However, as X can be made arbitrarily large, this assumption is without loss of generality.

Focusing on the S-MPE, we can get rid of the time subscript and to further simplify the notation, we denote by $x \in X$ the default policy for a given period with the understanding that $\bar{\gamma} = \{x, x\} \in X^2$.

For this model, S-MPE will be a combination of several components. For C, we are looking for four functions, two of them mapping x into the offered inflation in each period $p_D(x), p_A(x) : X \to X$ and the remaining two mapping x into the offered status-quo, $q_D(x), q_A(x) : X \to X$. Formally, we denote C's strategy $\rho_C = \{p_D(x), p_A(x), q_D(x), q_A(x)\} : X^4 \to X^4$.

For P, his strategy in each period maps combination of $\bar{\gamma}$ and γ into his vote. As his strategy will differ in D and A periods, the strategy is a mapping $\rho_P : X^8 \to \{yes, no\}$.

Notice that any given pair of strategies $\rho = \{\rho_C, \rho_P\}$ for a given x and a given path of D and A periods generates unique path of implemented inflation decisions $\{p_0, p_1, \ldots\}$. Taking expectations over all possible paths gives continuation value function for each policy-maker who knows x but does not know whether the next period will be D or A one,

$$V_C^\rho(x) = \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (p_t - \pi^* + \phi I_D(t))^2\right]$$
$$V_P^\rho(x) = \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (p_t - \pi^* - \phi I_D(t))^2\right]$$

where $I_D(t)$ is D-period indicator function and the superscript ρ captures dependence on given ρ. Having the continuation value functions we observe those can be equivalently derived as

$$V_C^\rho(x) = p_d\left[-(p_D^\rho(x) - \pi^* + \phi)^2 + \delta V_C^\rho(q_D^\rho(x))\right] + (1-p_d)\left[-(p_A^\rho(x) - \pi^*)^2 + \delta V_C^\rho(q_A^\rho(x))\right]$$
$$V_P^\rho(x) = p_d\left[-(p_D^\rho(x) - \pi^* - \phi)^2 + \delta V_P^\rho(q_D^\rho(x))\right] + (1-p_d)\left[-(p_A^\rho(x) - \pi^*)^2 + \delta V_P^\rho(q_A^\rho(x))\right].$$

Finally, we denote by $A_i(x)$ P's acceptance set in period $i \in \{A, D\}$ given a default policy x and strategies ρ by

$$A_D^\rho(x) = \{(p,q) \in X^2 | -(p - \pi^* - \phi)^2 + \delta V_P^\rho(q) \geq -(x - \pi^* - \phi)^2 + \delta V_P^\rho(x)\}$$
$$A_A^\rho(x) = \{(p,q) \in X^2 | -(p - \pi^*)^2 + \delta V_P^\rho(q) \geq -(x - \pi^*)^2 + \delta V_P^\rho(x)\}.$$

It is immediate that both of the acceptance sets are nonempty and compact.

With this notation, C's problem can be restated in terms of pair of the usual Bellman functional equations

$$\begin{aligned} U_D^\rho(x) &= \max_{\{p,q\} \in A_D^\rho(x)} \{-(p - \pi^* + \phi)^2 + \delta p_d U_D^\rho(q) + \delta(1-p_d)U_A^\rho(q)\} \\ U_A^\rho(x) &= \max_{\{p,q\} \in A_A^\rho(x)} \{-(p - \pi^*)^2 + \delta p_d U_D^\rho(q) + \delta(1-p_d)U_A^\rho(q)\} \end{aligned} \tag{2}$$

and the definition of S-MPE we use is

Definition 1 (Stationary Markov Perfect Equilibrium)**.** *A pair of strategies $\rho^* = \{\rho_C^*, \rho_P^*\}$ constitutes an S-MPE if for all $x \in X$ and any period $i \in \{A, D\}$*

1. *C's proposal strategy ρ_C^* solves (2)*
2. *P votes for C's proposal γ if and only if $\gamma \in A_i^{\rho^*}(x)$*
3. *In the model without the directive, C's strategies are restricted to offers $\{p, q\}$ satisfying $p = q$.*

Equivalent way to express the requirement of the S-MPE is to say we are looking for ρ giving rise to V_C^ρ and V_P^ρ such that when C and P maximize their utility in the current period their optimal behaviour is indeed expressed as ρ. If we can find such ρ then by the one deviation principle we have an equilibrium. From here on we focus on the equilibrium strategies and we drop the superscript ρ whenever the chance of confusion is minimal.

Equilibrium without the directive

Before proceeding further, let us make a weak assumption about the δ and p_d parameters in the model.

Assumption 1. *For any pair (δ, p_d) let $\delta^2 p_d(3-2p_d) \leq 1-\delta(1-p_d)$.*

Loosely speaking by making this assumption we are ruling out peculiar equilibria where C in D periods and for some default policies x offers inflation even higher than is the current x which goes against her contemporaneous incentives and she does so only to improve her bargaining position in the future. We regard this as an unrealistic feature and hence rule it out.

In terms of strictness of the assumption it can be expressed as $\delta \leq \varphi(p_d)$ where $\varphi(0) = \varphi(1) = 1$ and $\min_{p_d \in (0,1)} \varphi(p_d) = 7/9$ so that in effect we are ruling out equilibria where the 'future looms large' as δ approaches unity.

To characterize the equilibrium, we first conjecture that C's offers make P always indifferent between γ and $\bar{\gamma}$ provided C cannot implement her most preferred policy. Also, intuitively it should be the case that the set of default policies x for which C brings P to indifference in the A periods should be a subset of the policies for which C does the same in the D periods. Furthermore, building on the intuition from the two-period model, there should be a set of default policies for which the equilibrium offers are constant with respect to x as C can implement her most preferred policy. With this conjecture and some guess-work as to what are the appropriate intervals and what C's most preferred policy means, P's continuation value function can be shown to be

$$V_P(x) = \begin{cases} -\dfrac{1}{1-\delta}\left[(x-\pi^*-\phi p_d)^2 + \phi^2 p_d(1-p_d)\right] \\ \qquad \text{for } \; x \in \langle \pi^* - \phi\delta p_d, \pi^* + 3\phi\delta p_d\rangle \\ -\dfrac{p_d}{1-\delta p_d}\left[(x-\pi^*-\phi)^2 + \phi^2\dfrac{\delta(1-p_d)(1+3\delta p_d)}{1-\delta}\right] \\ \qquad \text{for } \; x \in \langle \pi^* - \phi, \pi^* - \phi\delta p_d\rangle \cup \langle \pi^* + 3\phi\delta p_d, \pi^* + 3\phi\rangle \\ -\dfrac{\phi^2 p_d}{1-\delta}(4-3\delta(1-p_d)) \\ \qquad \text{otherwise.} \end{cases} \tag{3}$$

The first part of V_P applies when C makes P indifferent between $\bar{\gamma}$ and γ in both, D and A periods. The second part applies when C makes P indifferent only in D periods but implements her most preferred policy in the A periods.

Finally, the third part applies when C can implement her most preferred policy in both, D and A periods. Notice also that V_P is continuous and differentiable on X except at the break points. With V_P pinned down, it can be shown that the equilibrium takes the following form.

Proposition 1 (S-MPE without the directive)**.** *Under the assumption 1 the equilibrium exists, is essentially unique, equilibrium offers are*

$$p_D(x) = q_D(x) = max\left\{min\{z \in X | z \in A_D(x)\}, \pi^* - \phi\right\}$$
$$p_A(x) = q_A(x) = max\left\{min\{z \in X | z \in A_A(x)\}, \pi^* - \phi\delta p_d\right\}$$

and P always accepts.

Proof. See appendix

In words, C always offers the lowest inflation she possibly can for both types of periods, provided she cannot reach her most preferred policy, which is $\pi^* - \phi$ in D periods and $\pi^* - \phi\delta p_d$ in A periods.

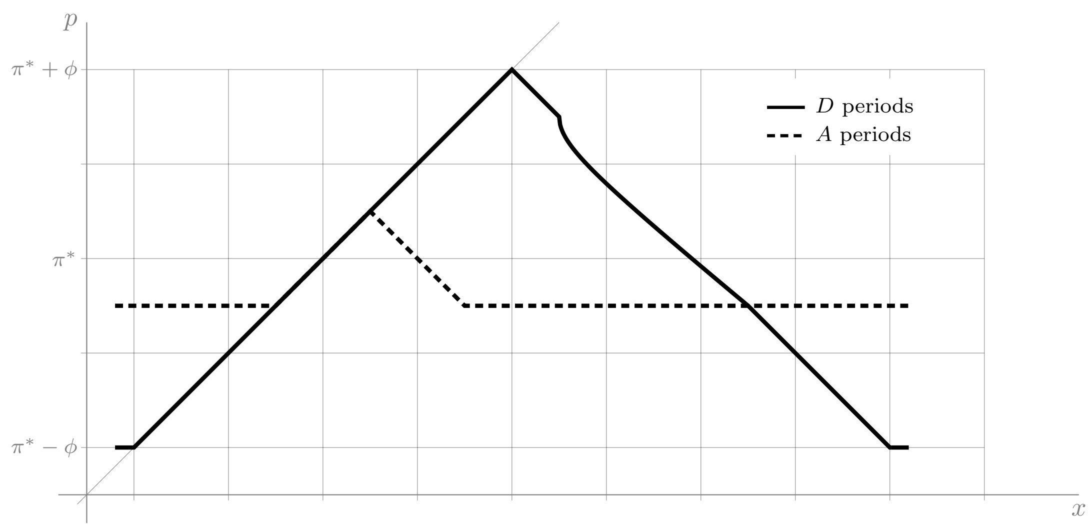
The strategy of the proof is following. We conjecture the V_P given above and the structure of the equilibrium given in proposition 1. Having done that, we derive C's continuation value function V_C and confirm she indeed want's to implement the minimum of P's acceptance sets when her overall optimum is not available.

To prove uniqueness, we take the P's acceptance correspondences A_D and A_A along with the V_P given in (3) and show that with those solution to (2) is unique using the extended version of the theorem 4.6 from Stokey and Lucas (1989) which we also prove. The essential adjective then means that we are proving uniqueness in the class of equilibria where P is brought to indifference whenever C cannot implement her optimal policy. However, we view this as a reasonable requirement on the equilibrium of the game where there is a conflict of interest between the two players.

To see how the equilibrium looks like in a graphical form, figure 1 shows particular parametrization for $\pi^* = 2, \phi = 1, \delta = 0.5, p_d = 0.5$. For all the equilibria of the model, the $p_A(x)$ offer policy looks exactly the same as on the figure. However, there are some differences regarding the shape of the $p_D(x)$ function. What is common to all of them is the constant and then linear increasing part for low values of x. Nevertheless, the x for which $p_D(x)$ reaches maximum in general differs depending on the parameters and the 'right' part of the $p_D(x)$ is not necessarily monotone or even continuous. One common feature is that it eventually decreases to $\pi^* - \phi$ where it becomes a constant function again.

In order to discuss the bargaining outcomes generated by the equilibrium, we find it helpful to define a set of x which, when reached, remains the offered and hence default policy for ever. We call this set of jointly absorbing default polices and define it along with the set of efficient default policies in the following definition.

Figure 1: Equilibrium policy without directive
$\pi^* = 2, \phi = 1, \delta = 0.5, p_d = 0.5$



Definition 2 (Set of jointly absorbing states and set of efficient states)**.**
A set $J \subseteq X$ defined as

$$J = \{x \in X | q_D(x) = q_A(x) = x\}$$

is called set of jointly absorbing states (jointly absorbing set).
A set $J_E \subseteq X$ defined as

$$J_E = \{x \in X | p_A(x) = \pi^*\}$$

is called set of efficient states (efficient set).

The rationale behind the definition of jointly absorbing set is that once the bargaining reaches $x \in J$ the resulting inflation and status-quo policy decisions are constant forever for any path of D and A periods. One interpretation of J is that it is the set of default policies for which the bargaining outcomes are irresponsive to the changing preferences of the policy-makers. However, it needs to be stressed that such an interpretation applies only to the model without the directive as $q_D(x) = q_A(x)$ implies $p_D(x) = p_A(x)$ which needs not hold in the model with the directive.

The rationale behind the notion of inefficient $X \setminus J_E$ set is that for any $x \in X \setminus J_E$ the policy-makers fail to agree on their current-period most preferred policy π^* due to their concerns about their bargaining position in the future. If in the A period with the default policy x they could sign a binding contract specifying that the next period default policy is x but today's inflation is π^*, both of them would be made better of.

Discussing the inflation outcomes is further complicated by the fact that those will in general depend on x with which bargaining starts and on the given path of A and D periods which is stochastic. Nevertheless, following proposition captures the key features.

Proposition 2 (Policy outcomes without the directive)**.**

1. *For any $x \in X$, the sequence of inflation decisions generated by x and any path of D and A periods reaches J in finite number of periods.*

2. *J has measure $2\phi\delta p_d$.*

3. *J_E has measure zero.*

Proof. See appendix

Recalling the equilibrium in figure 1 the intuition behind the result is straightforward. For any x in the A period inflation reaches J immediately and hence can stay out of J only for the path of D periods. And as the probability of n consecutive D periods goes to zero, inflation eventually falls into J. Part 2 of the proposition is immediately apparent from the figure realizing that the minimum of $p_A(x)$ is at $\pi^* - \phi\delta p_d$ and its maximum at $\pi^* + \phi\delta p_d$. Finally the last part is immediate from the picture.

What the proposition 2 is telling us is that in a S-MPE of the bargaining game without the directive, the inflation outcomes eventually become constant across periods at the level that does not necessarily corresponds to the inflation preferred by both policy-makers in the A periods.

Equilibrium with the directive

We now show how the bargaining outcomes change when C's offers are not restricted to those with equal inflation and status-quo. The first result we prove is that inflation in A periods is equal to π^* for any default policy. The logic behind the result is that since in the A periods the preferences of the policy-makers are aligned along the inflation dimension, there is no reason they should not be able to reach an agreement on the inflation set. And as changing inflation does not necessarily changes the status-quo for the next period, there is no trade-off for C to be made. The intuition is confirmed by the proposition.

Proposition 3 ($p_A(x)$ with the directive)**.** *Assume an equilibrium with the directive exists. Then for any $x \in X$*

$$p_A(x) = \pi^*$$

Proof. See appendix

Having established this result we are interested in the existence of the equilibrium for the model with the directive. Only then we can be sure that the bargaining protocol has the strong impact on the bargaining outcomes as suggested. Differently from the model without the directive where the existence followed by construction, situation is complicated by the fact that closed form solutions for the model with the directive are hard to obtain.

To establish the existence result, it is helpful first to pin down the V_P function and hence the shape of the acceptance correspondences A_D and A_A. Guided by the intuition, there should be three cases. First, for really high or really low default policies x, C should be able to implement her most preferred policy since it makes P still better off. Second, for the default policies that fall into the region of 'full conflict' between C and P, C by maximizing her utility should bring P to indifference between the default policy and her offer both in A and D periods. Finally, for the intermediate cases, the conflict between C and P should prevail in D periods but not in A periods. The intuition indeed turns out to be correct and allows us to pin down the V_P function.

Proposition 4 (V_P for the model with the directive)**.** *Assume an equilibrium with the directive exists. Then*

$$V_P(x) = \begin{cases} -\dfrac{1}{1-\delta}\left[(x-\pi^*-\phi p_d)^2+\phi^2 p_d(1-p_d)\right] \\ \qquad \textit{for} \;\; x \in \langle \pi^*+\phi\delta p_d-\kappa, \pi^*+\phi\delta p_d+\kappa\rangle \\ -\dfrac{p_d}{1-\delta p_d}\left[(x-\pi^*-\phi)^2+\phi^2\dfrac{4\delta(1-p_d)}{1-\delta}\right] \\ \qquad \textit{for} \;\; x \in \langle \pi^*-\phi, \pi^*+\phi\delta p_d-\kappa\rangle \cup \langle \pi^*+\phi\delta p_d+\kappa, \pi^*+3\phi\rangle \\ -\dfrac{4\phi^2 p_d}{1-\delta} \\ \qquad \textit{otherwise} \end{cases}$$

with $\kappa = \phi\sqrt{\delta p_d(3+\delta p_d)}$.

Proof. See appendix

Having established the shape of the V_P function, we are able to prove the upper-hemicontinuity of the acceptance correspondences for both types of periods. With this result we can finally prove existence of the equilibrium.

Proposition 5 (S-MPE with the directive)**.** *The equilibrium in the model with the directive exists and is essentially unique.*

Proof. See appendix

The idea of the proof is following. With the V_P given in proposition 4 we prove the upper-hemicontinuity of the acceptance correspondences A_D and A_A which again allows us to use the theorem used to prove the proposition 1. The essential uniqueness part comes from the fact that we are able to prove uniqueness of the resulting V_C function not of the resulting equilibrium offers.

Indeed when proving the proposition 3 we have shown that for $x \in X \setminus (\pi^* - \phi, \pi^* + 3\phi)$ the equilibrium offers are $q_D(x) = z$ and $q_A(x) = z'$ where $z, z' \in X \setminus (\pi^* - \phi, \pi^* + 3\phi)$. This somewhat complicates the characterization of J and J_E sets. However we are able to show the following result.

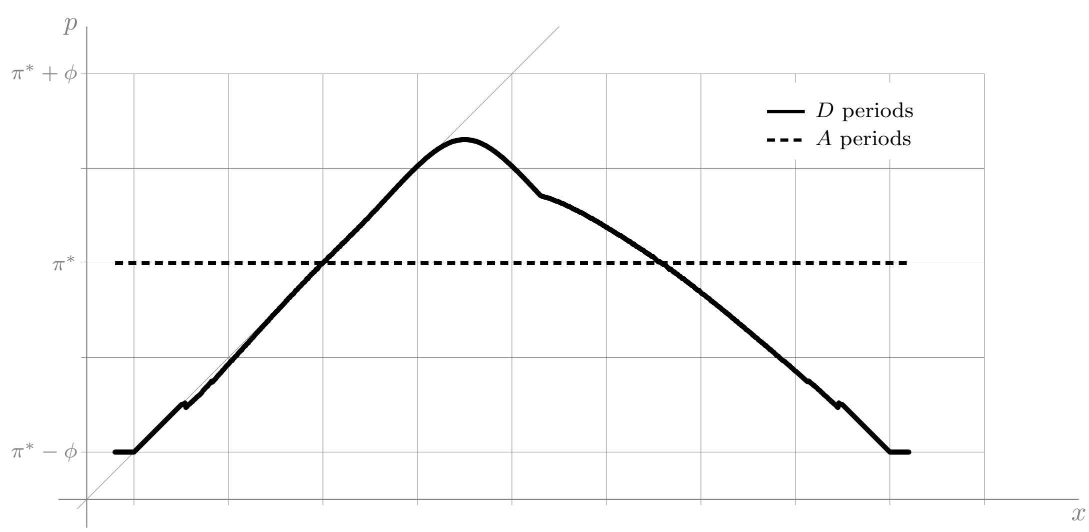
Proposition 6 (Policy outcomes with the directive)**.**

1. *J has at most the same measure as $X \setminus (\pi^* - \phi, \pi^* + 3\phi)$.*
2. *Subset of J where $p_D(x) = p_A(x)$ has measure zero.*
3. *J_E has the same measure as X.*

Proof. See appendix

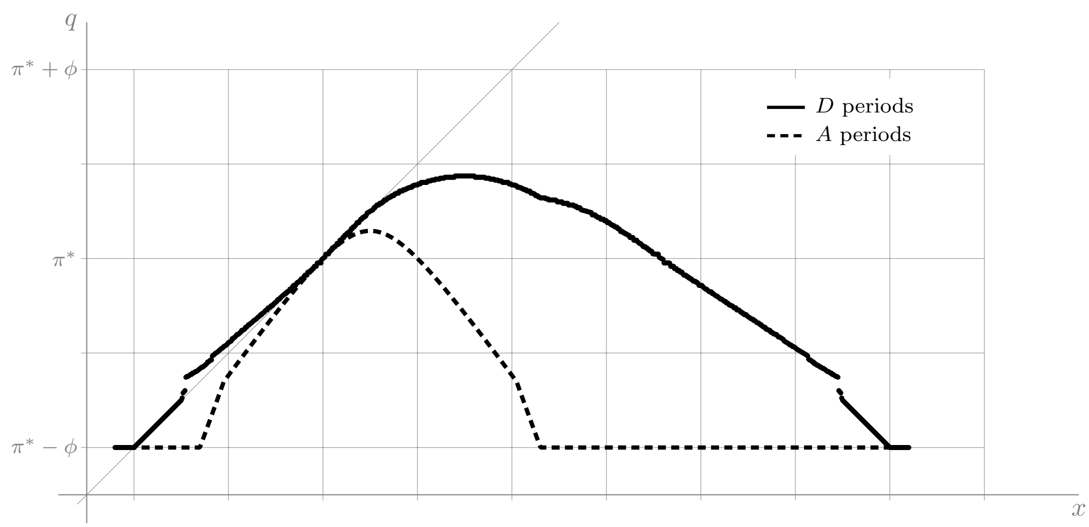
The most remarkable implication of the proposition 6 is the fact that the set of efficient states J_E is equal to the whole policy space X. We take this fact as an evidence of consensus building potential of the bargaining protocol with the directive.

Figure 2: Equilibrium inflation with directive
$\pi^* = 2, \phi = 1, \delta = 0.5, p_d = 0.5$



To see how the equilibrium offers look like, we next turn to their numerical estimation (see appendix for the details). We chose this route as the closed form solution to the C's optimization problem turns rather challenging to obtain. Figure 2 presents numerical estimation of the inflation proposals and figure 3 presents numerical estimation of the status-quo proposals. Where multiple offers

Figure 3: Equilibrium status-quo with directive
$\pi^* = 2, \phi = 1, \delta = 0.5, p_d = 0.5$



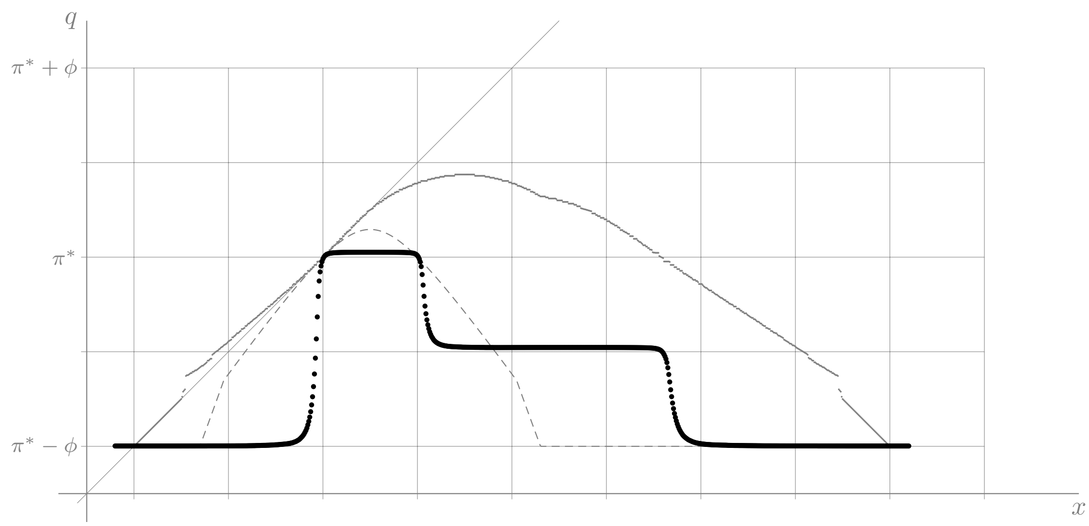
solve C's optimization problem we simply choose one of the optimal values, in practice always $\pi^* - \phi$. This is the case for the constant parts of the figure 3.

Looking at the figures 2 and 3 it is not immediately apparent whether the bargaining ever reaches a point where it would remain 'stable'. By the proposition 2 for the model without the directive the stable set J is reached in the finite number of periods. For the model with the directive we are not able to prove similar result as we do not have a closed form solution for the equilibrium offers.

To shed a light on this question we generated 10.000 one hundred period long random paths of A and D periods. For each path, we derived a last period status-quo offer by C as a function of the initial status-quo. Averaging over all the 10.000 paths gives the figure 4 which also depicts the equilibrium status-quo offers $q_D(x)$ and $q_A(x)$.

Looking at the figure, for the default policies x for which $q_D(x) < \pi^*$ and $q_A(x) < \pi^*$ holds, the bargaining over the long term converges to the status-quo of $\pi^* - \phi$. This is the case as C is able to improve on his bargaining position in A periods as she is happy to offer inflation equal to π^*. By offering inflation π^*, C makes P better off and uses this to offer status-quo that suits her preferences which means steering the status-quo in the direction of the set $X \setminus (\pi^* - \phi, \pi^* + 3\phi)$. This in hand implies that the constant part of the line in the figure 4 should be interpreted as $\pi^* - \phi$ or any other status-quo in $X \setminus (\pi^* - \phi, \pi^* + 3\phi)$. In terms of inflation outcomes this implies that in the

Figure 4: Long-run status-quo with the directive
average over 10.000 random 100 period long paths
$\pi^* = 2, \phi = 1, \delta = 0.5, p_d = 0.5$



long run inflation offers will be $\pi^* - \phi$ in the D periods and π^* in the A periods.

The convergence to $\pi^* - \phi$ also means that C gives up less of a bargaining position in D periods than it gains in the A periods. Intuitively, C gives up some of her bargaining position in the D periods as she trades-off the cost of doing so against the benefit of being able to offer inflation that is closer to her preferred point $\pi^* - \phi$. And she is happy to do so as she knows that she will be able to regain her bargaining position in the A period.

Notice also that for the status-quo policy $\pi^* - \phi$, C becomes dictator in the committee as she is always able to implement policies that fully reflect her preferences. We note similarity of this result with often mentioned dominant position of Chairman Greenspan in the FOMC (see for example Chappell et al. (2005) chapter 8). Also noteworthy is the fact that C has to build up the dominant position only gradually over time. More specifically, C improves on her bargaining position in every A period. But until the status-quo reaches $\pi^* - \phi$, she still has to take into account preferences of other committee members.

For the default policies x for which $q_D(x) > \pi^*$ and $q_A(x) > \pi^*$ holds, the status-quo converges to π^*. This is so as C is never able or willing to implement policy with status-quo that would start the convergent process to $\pi^* - \phi$ discussed above. In terms of policy outcomes consulting figure 2 shows that in the long term the inflation will be equal to π^* not only in A periods but also in D periods despite the diverging preferences of the committee members in the D periods.

Lastly for the default policies x for which $q_D(x) > \pi^*$ and $q_A(x) < \pi^*$ the long term outcome of the bargaining depends crucially on the first period. If the bargaining starts with A period, C is able to start the convergent process towards status-quo $\pi^* - \phi$ and eventually becomes dictator in the committee. Should the bargaining start with a D period, C's offer starts the convergence to status-quo π^* and the committee eventually reaches a position when it implements inflation equal to π^* in both types of periods. The line between π^* and $\pi^* - \phi$ then reflects the fact that proportion p_d of the paths eventually converges to $\pi^* - \phi$ whereas the remaining paths converge to π^*.

Notice the strong path dependency displayed by the model. For some default policies x the committee eventually becomes dominated by its chairman. For some default policies the committee becomes 'consensual' in that the inflation implemented in a disagreement periods is midway between the inflations preferred by its members. Finally, for some default policies the first period plays a crucial role and determines whether the committee becomes chairman dominated or consensual.

Multi-member committee and further comments

One obvious objection to the model presented in this paper is the fact that typical monetary committee is composed of more than two members. What we want to know is whether the results presented remain valid when we add another members along the C and P.

However, there are many ways how to expand the two person committee in terms of the resulting preference structures. For this reason we focus on a relatively simple committee expansion process we call median preserving and define as follows.

Definition 3 (Median preserving committee expansion)**.** *We say a committee is expanded in the median preserving way if in the initial step member with the preference parameter $\phi_0 > \phi$ is added and in arbitrary number N' of subsequent steps each expansion $n \leq N'$ adds pair of members with $\phi_{n,1} > \phi$ and $\phi_{n,2} < \phi$.*

In words, in the first step of the expansion we add a member who is more extreme than P in that his preference parameter ϕ_0 satisfies $\phi_0 > \phi$. Notice this steps makes P the 'true' median member as the number of committee members becomes odd.

For any number of subsequent expansion steps, we then require members to be added in pairs in each step n in order to preserve the odd-member feature and restrict the additional members to have preference parameters $\phi_{n,1}$ and $\phi_{n,2}$ that satisfy $\phi_{n,1} > \phi > \phi_{n,2}$. It is easy to confirm that P remains median member after any number of such expansions and final size of the committee is $3 + 2N' = N$.

Next we need to rule out equilibria which possibly arise due to the committee members voting against their preferences as they realize they are not pivotal. Following Baron and Kalai (1993) we restrict attention to *stage-undominated* voting strategies that for all members $n \in N$, all periods $i \in \{A, D\}$, all default policies $x \in X$ and all proposals $\gamma \in X^2$ satisfy

$$n \text{ votes } \textit{yes} \text{ for } \gamma \Leftrightarrow \gamma \in A_{i,n}(x).$$

With the preliminaries established, we are able to prove the following proposition asserting that the results presented so far can be equally applied to any larger committee.

Proposition 7 (Committee with more than 2 members)**.** *Expanding the committee of the models above in a median preserving way and assuming members use stage-undominated voting strategies leaves all the results unchanged.*

Proof. See appendix

Another interesting question arises from comparison of the two bargaining structures. Assume that C and P, before starting the game analyzed in this paper and most importantly before the first default policy is determined, have an option to choose between the bargaining protocols. Would they prefer either of the protocols and does it depend on their believes about the first default policy?

Figure 5: Equilibrium value functions
$\pi^* = 2, \phi = 1, \delta = 0.5, p_d = 0.5$

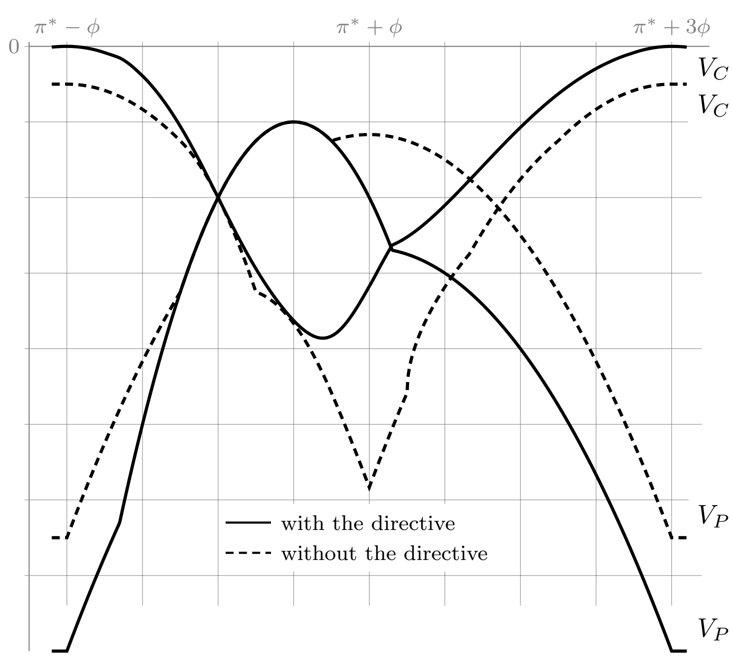


Figure 5 illustrates the answer to this question. It depicts value functions of both policy-makers for the two bargaining protocols. All the functions are based on the analytical results except for the V_C function in the model with the directive which comes from the simulation exercise.

Note that the first intuition that the bargaining protocol with the directive would be preferred as it relaxes the constraint on the C's optimization problem is misleading as it does not take into account change in P's strategic behaviour. This argument would indeed be correct if we could prove that the P's acceptance sets without the directive are subsets of the acceptance sets with the directive.

However, this turns out not to be true and it is relatively easy to construct examples where P's acceptance sets without the directive include policies which are not included in the acceptance sets without the directive. It follows C might be worse off for some default policies with the directive. Hence the bargaining protocol she prefers will in general depend on her believes about the initial default policy.

For P, the question is less complicated, not least as we have explicit expressions for V_P. It turns out P (weakly) prefers the bargaining protocol without the directive.

The intuition behind the result is that for the default policies for which he is made indifferent between γ and $\bar{\gamma}$, his continuation value is equal under the two bargaining protocols. At the same time for the default policies for which C is able to extract all the bargaining power over the long periods under the bargaining with the directive, P prefers the bargaining protocol without the directive. This is so because under this bargaining protocol he retains some influence over the enacted policies which then reflect, at least to some extent, his preferences.

Finally we were interested whether the model with the directive generates bargaining outcomes mimicking the real world ones. More specifically, we focus on one of the arguments used to support the notion that the directive serves consensus building role. As Thornton and Wheelock (2000) note, FOMC meetings during which target federal funds rate remains unchanged predominantly adopt asymmetric directive. At the same time symmetric directive is usually adopted during meetings when the rate is changed.

To check whether our model is able to deliver the same prediction, we generated 100.000 random two period long paths each of them for random initial default policy drawn from the interval $\langle \pi^* - \phi, \pi^* + \phi \rangle$. For each path, we first recorded the inflation and status quo adopted in the first period, p_1 and q_1. We then moved to the second period for which the default policy was q_1 and we recorded resulting inflation and status-quo, p_2 and q_2.

Having done that, we coded second period on each path according to two criteria. First criterium was whether change in inflation compared to the first period took place. Periods which satisfied $|p_2 - p_1| \leq \varepsilon$ were coded as *no change* ones. Second criterium was whether the committee adopted symmetric directive or not. In this respect, periods which satisfied $|q_2 - p_2| \leq \varepsilon$ were coded as *symmetric*. Table 1 gives results of our exercise along with those taken from Thornton and Wheelock (2000) page 10.

Table 1: Ratio of asymmetric to symmetric directives

	rate change	no rate change
Thornton and Wheelock	0.60	1.71
Model	≈ 0.64	≈ 1.26

With ε chosen to match the ratio for the change meetings, the model somewhat under-predicts number of asymmetric directives in the no change meetings ($\approx$ sign is meant to highlight that the results slightly differ for each run). Nevertheless, it correctly generates mostly asymmetric directives for the change and mostly symmetric directives for the no change meetings. However, what is not apparent from the table is that the model generates too few no change meetings in general, but this is not surprising given that the FOMC usually decides in a discrete steps whereas the policy space the model assumes is continuous.

6 Conclusion

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References

Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis, A Hitchhiker's Guide*. Springer, Berlin, 2006.

David P. Baron and Ehud Kalai. The simplest equilibrium of a majority-rule division game. *Journal of Economic Theory*, 61(2):290–301, 1993.

Robert J. Barro and David B. Gordon. Rules, discretion and reputation in a model of monetary policy. *Journal of Monetary Economics*, 12(1):101–121, 1983.

Sterling K. Berberian. *Introduction to Hilbert Space*. American Mathematical Society, New York, 1999.

Alan S. Blinder. *Central Banking in Theory and Practice*. MIT Press, London, 1998.

Alan S. Blinder and John Morgan. Are two heads better than one?: An experimental analysis of group vs. individual decisionmaking. *National Bureau of Economic Research Working Paper Series*, No. 7909, 2000.

Nicolas Bourbaki. *Elements de Mathematique, Topologie Generale*. Springer, Berlin, 2007.

Kelly H. Chang. *Appointing Central Bankers: The Politics of Monetary Policy in the United States and the European Monetary Union*. Cambridge University Press, New York, 2003.

Henry W. Chappell, Rob R. McGregor, and Todd A. Vermilyea. *Committee Decisions on Monetary Policy, Evidence from Historical Records of the Federal Open Market Committee*. MIT Press, London, 2005.

Henry W. Chappell, Rob R. McGregor, and Todd A. Vermilyea. The role of the bias in crafting consensus: FOMC decision making in the Greenspan era. *International Journal of Central Banking*, 3(2):39–60, 2007.

Richard Cothren. Equilibrium inflation as determined by a policy committee. *The Quarterly Journal of Economics*, 103(2):429–434, 1988.

Ernesto Dal Bo. Committees with supermajority voting yield commitment with flexibility. *Journal of Public Economics*, 90(4-5):573–599, 2006.

Allan Drazen. *Political Economy in Macroeconomics*. Princeton University Press, New Jersey, 2000.

Michael Ehrmann and Marcel Fratzscher. Transparency, disclosure and the Federal Reserve. *International Journal of Central Banking*, 3(1):179–225, 2007.

Rasmus Fatum. One monetary policy and 18 central bankers: The European monetary policy as a game of strategic delegation. *Journal of Monetary Economics*, 53(4):659–669, 2006.

Petra Gerlach-Kristen. Monetary policy committees and interest rate setting. *European Economic Review*, 50(2):487–507, 2006.

Finn E. Kydland and Edward C. Prescott. Rules rather than discretion: The inconsistency of optimal plans. *Journal of Political Economy*, 85(3):473, 1977.

John S. Lapp and Douglas K. Pearce. Does a bias in FOMC policy directives help predict intermeeting policy changes? *Journal of Money, Credit and Banking*, 32(3):435–441, 2000.

Philipp Maier. Monetary policy committees in action: Is there a room for improvement? *Bank of Canada Working Paper*, No. 6, 2007.

Egil Matsen and Oistein Roisland. Interest rate decisions in an asymmetric monetary union. *European Journal of Political Economy*, 21(2):365–384, 2005.

Ellen E. Meade. The FOMC: Preferences, voting, and consensus. *The Federal Reserve Bank of St. Louis Review*, 87(2):93–101, 2005.

Ellen E. Meade and Nathan D. Sheets. Regional influences on FOMC voting patterns. *Journal of Money, Credit and Banking*, 37(4):661–677, 2005.

Laurence H. Meyer. *A Term at the Fed: An Insider's View*. Collins, New York, 2004.

Ilian Mihov and Anne Sibert. Credibility and flexibility with independent monetary policy committees. *The Journal of Money, Credit and Banking*, 38(1): 23–46, 2006.

Michael R. Pakko. On the information content of asymmetric FOMC policy statements: Evidence from a Taylor-rule perspective. *Economic Inquiry*, 43 (3):558–569, 2005.

Torsten Persson and Guido Tabellini. *Political Economics: Explaining Economic Policy*. MIT Press, London, 2000.

Alessandro Riboni. Committees as substitutes for commitment. *forthcoming in International Economic Review*, 2009.

Alessandro Riboni and Francisco J. Ruge-Murcia. The dynamic (in)efficiency of monetary policy by committee. *Journal of Money, Credit and Banking*, 40 (5):1001–1032, 2008.

Anne Sibert. Monetary policy committees: Individual and collective reputations. *Review of Economic Studies*, 70(3):649–665, 2003.

Anne Sibert. Central banking by committee. *International Finance*, 9(2):145–168, 2006.

Nancy L. Stokey and Robert E. Lucas. *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, MA, 1989.

Daniel L. Thornton and David C. Wheelock. A history of the asymmetric policy directive. *The Federal Reserve Bank of St. Louis Review*, 82(5):1–16, 2000.

Christopher J. Waller. Policy boards and policy smoothing. *The Quarterly Journal of Economics*, 115(1):305–339, 2000.

Christopher J. Waller. Monetary policy games and central bank politics. *Journal of Money, Credit and Banking*, 21(4):422–431, 1989.

Christopher J. Waller. A bargaining model of partisan appointments to the central bank. *Journal of Monetary Economics*, 29(3):411–428, 1992.

Appendices

A1 Proof of proposition 1

Preliminaries

To proof the proposition 1 we are forced to split the equilibria of the model without the directive into four distinct cases depending on parameters δ and p_d. However, the logic of the proof is always the same. We state the equilibrium offers and confirm they correspond to those given in the proposition using the shape of the induced V_P function. We then confirm by investigating the shape of induced V_C that C indeed wants to implement the minimum of the P's acceptance set or her overall optimum. Throughout the proof we always assume assumption 1 holds.

Despite the logic of the proof being rather straightforward, the proof itself is rather lengthy and algebra intensive. Striving to keep its length at minimum, we sometimes omit proofs of purely algebraic results but always indicate how those can be shown.

Throughout the proof, we often refer to C in D periods as to CD and similarly for P (PD) and by analogy in A periods to CA and PA respectively. To save on notation we denote current utility of the policy makers by

$$
\begin{aligned}
f_{CD}(x) &= -(x - \pi^* + \phi)^2 \qquad & f_{PD}(x) &= -(x - \pi^* - \phi)^2 \\
f_{CA}(x) &= -(x - \pi^*)^2 \qquad & f_{PA}(x) &= -(x - \pi^*)^2
\end{aligned}
$$

and the overall utility by

$$
\begin{aligned}
U_{CD}(x) &= f_{CD}(x) + \delta V_C(x) \qquad & U_{PD}(x) &= f_{PD}(x) + \delta V_P(x) \\
U_{CA}(x) &= f_{CA}(x) + \delta V_C(x) \qquad & U_{PA}(x) &= f_{PA}(x) + \delta V_P(x).
\end{aligned}
$$

Throughout the proof we are forced to work with series of intervals in the policy space. Those are always denoted by I_i and are always closed (except where explicitly indicated) and convex subsets of the policy space. The upper border of I_i is then denoted by I_i^U and lower border by I_i^L.

Many of the functions in the proof are defined piecewise. If this is the case then we use notation $f^{I_i}(x)$ for function $f(x)$ constrained to the appropriate interval. Derivatives are often denoted by primes when no confusion as to with respect to which variable the derivative is being taken is imminent.

It will become apparent that many of the functions we work with are differentiable only in the interior of the intervals but not at the point where the two intervals meet. Taking general $f(x)$, $f'(I_i^U)$ will often fail to exist as $f(x)$ has kink at I_i^U. If this is the case then $f'^{I_i}(I_i^U)$ will always denote left derivative, i.e. derivative as $x \to I_i^U$ from below, and $f'^{I_i}(I_i^L)$ will denote right derivative, i.e. derivative as $x \to I_i^L$ from above.

It is helpful first to establish following lemmas.

Lemma 1.

$$\begin{array}{ll} U'_{CD}(x) \geq 0 \Rightarrow U'_{CA}(x) \geq 0 & U'_{PD}(x) \geq 0 \Leftarrow U'_{PA}(x) \geq 0 \\ U'_{CD}(x) \leq 0 \Leftarrow U'_{CA}(x) \leq 0 & U'_{PD}(x) \leq 0 \Rightarrow U'_{PA}(x) \leq 0 \end{array}$$

Proof. Lemma follows from the readily verifiable facts that $f'_{CA}(x) > f'_{CD}(x)$ and $f'_{PA}(x) < f'_{PD}(x)$. $\square$

Lemma 2. *Let $h(x)$ and $k(x)$ be two real valued continuously differentiable functions defined on $\langle t-r, t\rangle$ and $\langle t, t+r\rangle$ respectively, for some $t, t, r \in \mathbb{R}$ and $r > 0$. Assume $k(t) = h(t)$ and that the first derivative of the functions satisfies $k'(t+x) \leq -h'(t-x)$ for all positive $x \leq r$. Then $k(t+r) \leq h(t-r)$.*

Proof. Integrating the derivative inequality in the lemma with respect to x from 0 to r gives

$$\begin{aligned} \int_0^r k'(t+z)dz &\leq -\int_0^r h'(t-z)dz \\ k(t+r) - k(t) &\leq h(t-r) - h(t) \\ k(t+r) &\leq h(t-r) \end{aligned}$$

$\square$

Lemma 3. *Define*

$$z(x) = \pi^* + \phi(1-\delta(1-p_d)) - \sqrt{\frac{1-\delta}{1-\delta p_d}(x-\pi^*-\phi)^2 + \phi^2\delta(1-p_d)\left(\frac{4\delta^2 p_d^2}{1-\delta p_d} - (1-\delta)\right)}.$$

Then

$$\begin{aligned} \operatorname{sgn}[z(x)'] &= \operatorname{sgn}[\pi^* + \phi - x] \\ \operatorname{sgn}[z(x)''] &= \operatorname{sgn}[-(4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d))]. \end{aligned}$$

Proof. Denote the term in the square root of $z(x)$ by $T(x)$. Then

$$\begin{aligned} z(x)' &= -\frac{1}{\sqrt{T(x)}}\frac{1-\delta}{1-\delta p_d}(x-\pi^*-\phi) \\ z(x)'' &= -\frac{1}{T(x)^{3/2}}\frac{1-\delta}{1-\delta p_d}\phi^2\delta(1-p_d)(4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d)). \end{aligned}$$

$\square$

Next we give explicit formulas for the continuation value functions of the two policy-makers used throughout the proof. As already mentioned, both of the functions are defined piecewise on the different I_i intervals, but we leave the specific definition of the intervals for later when we will show that in the equilibrium induced continuation value function of C can be put together from the following.

$$
\begin{aligned}
V_C^{I_1}(x) =& V_C^{I_{12}}(x) = -\frac{1-p_d}{1-\delta}\phi^2\delta p_d \\
V_C^{I_2}(x) =& V_C^{I_5}(x) = -\frac{p_d}{1-\delta p_d}\left[(x-\pi^*+\phi)^2+\phi^2\frac{\delta(1-p_d)(1-\delta p_d)}{1-\delta}\right] \\
V_C^{I_3}(x) =& -\frac{1}{1-\delta}\left[(x-\pi^*+\phi p_d)^2+\phi^2 p_d(1-p_d)\right] \\
V_C^{I_4}(x) =& V_C^{I_3}(x)+\frac{8(1-p_d)\delta p_d}{(1-\delta)(1-\delta p_d)}[\phi(x-\pi^*)-\phi^2\delta p_d] \\
V_C^{I_6}(x) =& V_C^{I_{11}}(x) = -\frac{p_d}{1-\delta p_d}\left[(\pi^*+3\phi-x)^2+\phi^2\frac{\delta(1-p_d)(1-\delta p_d)}{1-\delta}\right] \\
V_C^{I_7}(x) =& p_d\left[(2(\pi^*+\phi(1-\delta(1-p_d)))-x-\pi^*+\phi)^2+\delta V_C^{I_4}(2(\pi^*+\phi(1-\delta(1-p_d)))-x)\right] \\
& (1-p_d)\left[(2(\pi^*+\phi\delta p_d)-x-\pi^*)^2+\delta V_C^{I_3}(2(\pi^*+\phi\delta p_d)-x)\right] \\
V_C^{I_8}(x) =& p_d\left[(2(\pi^*+\phi(1-\delta(1-p_d)))-x-\pi^*+\phi)^2+\delta V_C^{I_3}(2(\pi^*+\phi(1-\delta(1-p_d)))-x)\right] \\
& (1-p_d)\left[(2(\pi^*+\phi\delta p_d)-x-\pi^*)^2+\delta V_C^{I_3}(2(\pi^*+\phi\delta p_d)-x)\right] \\
V_C^{I_9}(x) =& p_d\left[-(z(x)-\pi^*+\phi)^2+\delta V_C^{I_4}(z(x))\right]+(1-p_d)\left[-(-\phi\delta p_d)^2+\delta V_C^{I_3}(\pi^*-\phi\delta p_d)\right] \\
V_C^{I_{10}}(x) =& p_d\left[-(z(x)-\pi^*+\phi)^2+\delta V_C^{I_3}(z(x))\right]+(1-p_d)\left[-(-\phi\delta p_d)^2+\delta V_C^{I_3}(\pi^*-\phi\delta p_d)\right]
\end{aligned}
$$

Likewise, P's continuation value function in the equilibrium will be put together from the following functions.

$$
\begin{aligned}
V_P^{I_3}(x) =& -\frac{1}{1-\delta}\left[(x-\pi^*-\phi p_d)^2+\phi^2 p_d(1-p_d)\right] \\
&= V_P^{I_4}(x) = V_P^{I_7}(x) = V_P^{I_8}(x) \\
V_P^{I_2}(x) =& -\frac{p_d}{1-\delta p_d}\left[(x-\pi^*-\phi)^2+\phi^2\frac{\delta(1-p_d)(1+3\delta p_d)}{1-\delta}\right] \\
&= V_P^{I_5}(x) = V_P^{I_6}(x) = V_P^{I_9}(x) = V_P^{I_{10}}(x) = V_P^{I_{11}}(x) \\
V_P^{I_1}(x) =& V_P^{I_{12}}(x) = -\frac{\phi^2 p_d}{1-\delta}(4-3\delta(1-p_d))
\end{aligned}
$$

At the time being use of 12 different I_i's might seem redundant, but as will become apparent the fact that the value functions are identical on some intervals is a coincidence. Indeed, they will be induced by parts of the equilibrium that are different in nature.

Having the V_P function we can also explain a rationale behind the $z(x)$ from lemma 3. Looking at the V_P it is apparent that it consists of two quadratic equations which apply at the different I_i intervals. The $z(x)$ function then allows us to go from one of the quadratic equations to the other. In other words, $z(x)$ ensures $V_P(x) = V_P(z(x))$. More specifically, as the proposition claims that C implements policy corresponding to the minimal accepted one, $z(x)$ gives us lower border of the acceptance set for some default policy x.

We sometimes need to use an inverse of $z(x)$ as well. Formally speaking, as $z(x)$ is not monotone, $z^{-1}(x)$ is not well defined. However, it is apparent there are exactly two solutions x to the equation $k = z(x)$ for a given constant k. Taking the larger of the two, we can define inverse of the function $z(x)$ as $z^{-1}(x) = \{\max\{y : x = z(y)\}\}$.

Case 1: Equilibrium for $\delta \leq \frac{1}{1+2p_d}$

For $\delta \leq \frac{1}{1+2p_d}$ the equilibrium offers are

$$p_A(x) = \begin{cases} \pi^* - \phi\delta p_d & \text{for } x \in I_1 \cup I_2 \cup I_5 \cup I_6 \cup I_9 \cup I_{10} \cup I_{11} \cup I_{12} \\ x & \text{for } x \in I_3 \\ 2(\pi^* + \phi\delta p_d) - x & \text{for } x \in I_4 \end{cases}$$

$$p_D(x) = \begin{cases} \pi^* - \phi & \text{for } x \in I_1 \cup I_{12} \\ x & \text{for } x \in I_2 \cup I_3 \cup I_4 \cup I_5 \\ 2(\pi^* + \phi) - x & \text{for } x \in I_6 \cup I_{11} \\ z(x) & \text{for } x \in I_9 \cup I_{10} \end{cases}$$

where

$$\begin{aligned} I_1 &= \langle x^-, \pi^* - \phi\rangle & I_6 &= \langle \pi^* + \phi, \pi^* + \phi(2 - 3\delta p_d)\rangle \\ I_2 &= \langle \pi^* - \phi, \pi^* - \phi\delta p_d\rangle & I_9 &= \langle \pi^* + \phi(2 - 3\delta p_d), \tau^+\rangle \\ I_3 &= \langle \pi^* - \phi\delta p_d, \pi^* + \phi\delta p_d\rangle & I_{10} &= \langle \tau^+, \pi^* + \phi(2 + \delta p_d)\rangle \\ I_4 &= \langle \pi^* + \phi\delta p_d, \pi^* + 3\phi\delta p_d\rangle & I_{11} &= \langle \pi^* + \phi(2 + \delta p_d), \pi^* + 3\phi\rangle \\ I_5 &= \langle \pi^* + 3\phi\delta p_d, \pi^* + \phi\rangle & I_{12} &= \langle \pi^* + 3\phi, x^+\rangle \end{aligned}$$

where $\tau^+ = \pi^* + \phi + \phi\sqrt{(1 - \delta p_d)^2 - \frac{4\delta^3 p_d^2(1-p_d)}{1-\delta}}$ (τ^- to be used later is defined analogously with the term in the square root subtracted) and x^- and x^+ are respectively lower and upper border of the policy space X.

To see the term in the square root of τ^+ is always positive, substitute in $\delta = 1/(1 + 2p_d)$ which gives a positive expression. Then differentiating the original expression with respect to δ one gets an expression which can be regarded as cubic equation in δ. Upon solving it has one real root and the derivative is negative below the root. As the root is always higher than unity, it follows the original expression has to be positive.

It is straightforward to show that the equilibrium offers induce the continuation value functions given above on the appropriate I_i intervals and that both V_C and V_P are continuous everywhere and differentiable everywhere except at the borders of the I_i intervals. Next we need to pin down the shape of U_{PA} and U_{PD} functions.

claim 1 (Shape of U_{PA} and U_{PD}). *U_{PA} is increasing on $I_1 \cup I_2 \cup I_3$ and decreasing otherwise. U_{PD} is increasing on $I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$ and decreasing otherwise. U_{PA} has global maximum at $\pi^* + \phi\delta p_d$, U_{PD} has global maximum at $\pi^* + \phi$ and both functions are quasi-concave.*

Proof. It is straightforward to show that U_{PA} is increasing (and hence U_{PD} as well by lemma 1) on $I_1 \cup I_2 \cup I_3$. Similarly U_{PD} is decreasing (and hence U_{PA} by the same lemma) on $I_6 \cup I_9 \cup I_{10} \cup I_{11} \cup I_{12}$. The remaining two intervals, I_4 and I_5, are easy to show as well. It follows U_{PA} has to have global maximum at $\pi^* + \phi\delta p_d$ which is border of I_3 with I_4 and U_{PD} has to have global maximum at $\pi^* + \phi$ which is border of I_5 with I_6. Quasi-concavity then follows. □

Next two claims outline the shape of P's acceptance sets.

claim 2 (Shape of $A_A(x)$). *Let x be the default policy. Then*

1. *if $x \in I_3$ then $A_A(x) = \{p : x \leq p \wedge p \leq x'\}$ with $x' = 2(\pi^* + \phi\delta p_d) - x \in I_4$*
2. *if $x \in I_4$ then $A_A(x) = \{p : x' \leq p \wedge p \leq x\}$ with $x' = 2(\pi^* + \phi\delta p_d) - x \in I_3$*
3. *if $x \notin I_3 \cup I_4$ then $\pi^* - \phi\delta p_d \in A_A(x)$.*

Proof. Notice U_{PA} is symmetric around $\pi^* + \phi\delta p_d$ which is its global maximum on $I_3 \cup I_4$. Moreover for any $x \in I_3$, U_{PA} is increasing up to x and for any $x \in I_4$, U_{PA} is decreasing from x on. Hence the first part follows. Similar argument proves the second part.

To see the third part, notice $U_{PA}(I_3^L) = U_{PA}(I_4^U)$ and $I_3^L = \pi^* - \phi\delta p_d$. The third part then follows by the same argument as in the preceding paragraph about the increasing and decreasing parts of U_{PA}. □

claim 3 (Shape of $A_D(x)$). *Let x be the default policy. Then*

1. *if $x \in I_1 \cup I_{12}$ then $\pi^* - \phi \in A_D(x)$*
2. *if $x \in I_2$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = 2(\pi^* + \phi) - x \in I_{11}$*
3. *if $x \in I_3 \cup I_4$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = z^{-1}(x) \in I_9 \cup I_{10}$*
4. *if $x \in I_5$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = 2(\pi^* + \phi) - x \in I_6$*
5. *if $x \in I_6$ then $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ where $x' = 2(\pi^* + \phi) - x \in I_5$*
6. *if $x \in I_9 \cup I_{10}$ then $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ where $x' = z(x) \in I_3 \cup I_4$*
7. *if $x \in I_{11}$ then $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ where $x' = 2(\pi^* + \phi) - x \in I_2$.*

Proof. All the parts below use the fact that for $x \leq \pi^* + \phi$, U_{PD} is increasing up to x and for $x \geq \pi^* + \phi$, U_{PD} is decreasing from x on. Also convexity of $A_D(x)$ for given x follows from quasi-concavity of U_{PD}.

For part one, notice $U_{PD}(I_1^U) = U_{PD}(I_{12}^L)$ and $I_1^U = \pi^* - \phi$ which along with the argument in the preceding paragraph gives the result.

For part two, notice U_{PD} is symmetric around $\pi^* + \phi$ for $x \in I_2 \cup I_{11}$. This also proves part seven.

For part three, by quasi-concavity of U_{PD} and the fact that U_{PD} has global maximum at $\pi^* + \phi$ there must exist upper border of the acceptance set which satisfies $x' \geq \pi^* + \phi$. It is easy to confirm $x' \in I_9 \cup I_{10}$ and that x' has to solve $x = z(x')$, i.e. $x' = z^{-1}(x)$.

For part four, notice U_{PD} is symmetric around $\pi^* + \phi$ for $x \in I_5 \cup I_6$. Hence the fourth part follows. This also proves part five.

For part six, we are looking for x' which solves $U_{PD}(x) = U_{PD}(x')$ with $x \in I_9 \cup I_{10}$. It is easy to confirm $x' = z(x) \in I_3 \cup I_4$ is the solution to this equation. □

Following claim gives the shape U_{CD} and U_{CA} functions.

claim 4 (Shape of U_{CA} and U_{CD})**.**

1. *U_{CA} is increasing on $I_1 \cup I_2$ and decreasing on $I_3 \cup I_5 \cup I_6 \cup I_{10} \cup I_{11} \cup I_{12}$*
2. *U_{CD} is increasing on I_1 and decreasing on $I_2 \cup I_3 \cup I_4 \cup I_5 \cup I_6 \cup I_{10} \cup I_{11} \cup I_{12}$*
3. *$U_{CA}(x) \geq U_{CA}(x')$ where $x \in I_3$ and $x' = 2(\pi^* + \phi\delta p_d) - x \in I_4$*
4. *$U_{CA}(\pi^* - \phi\delta p_d) \geq \max_{x \in I_9} U_{CA}(x)$*
5. *$U_{CD}(z(x)) \geq U_{CD}(x')\ \forall x' \in \langle I_9^L, x\rangle$ given $x \in I_9$*
6. *U_{CA} has global maximum at $\pi^* - \phi\delta p_d$ and U_{CD} at $\pi^* - \phi$.*

Proof. The first part is straightforward given the continuation value functions above, except for I_{10}. To establish $U_{CA}^{I_{10}}$ is decreasing, first note

$$V_C''^{I_{10}}(x) = p_d z(x)'' \left[U_{CD}'^{I_3}(z(x)) \right] - p_d \frac{2}{1-\delta} [z(x)']^2.$$

Sign of $z(x)''$ by lemma 3 depends on sign of $4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d)$ which is negative for $\delta \leq 1/(1+2p_d)$ and hence $z(x)''$ is positive. Sign of $U_{CD}'^{I_3}(z(x))$ is negative by the part two of this claim and the last term is negative so the $V_C''^{I_{10}}(x)$ is negative. It follows $U_{CA}''^{I_{10}}$ is concave so if we can establish that $U_{CA}'^{I_{10}}(I_{10}^L)$ is negative the claim follows.

Evaluating $U_{CA}'^{I_{10}}(x)$ at $I_{10}^L = \tau^+$ gives

$$U_{CA}'^{I_{10}}(\tau^+) = -2\phi \left[1 + \left(\frac{\tau^+ - \pi^* - \phi}{\phi} \right) \left(\frac{1 - \delta - 2\delta p_d (1 - \delta(1-p_d))}{(1-\delta)(1-\delta p_d)} \right) \right]$$

where the term in the brackets is positive. To see this, note that the last term in the equation $1 - \delta - 2\delta p_d(1 - \delta(1-p_d)) > 0$. This can be seen regarding the expression as a quadratic equation in δ. It is negative between the roots. One of the roots is higher than unity and the second one is higher than $1/(1+2p_d)$. This establishes the first part.

For the second part, it is again straightforward to establish most of the results. For I_{10} the claim follows from the part one of this claim and lemma 1 and for I_4 the claim follows by assumption 1.

The third part follows readily from the derivatives of U_{CA} on I_3 and I_4 using lemma 2 which can be used as I_3 and I_4 have the same width.

To establish the fourth part where we cannot use the derivative argument as U_{CA} may have local maximum on I_9, first note

$$V_C'^{I_9}(x) = p_d z(x)' \left[U_{CD}'^{I_4}(z(x)) \right]$$

which by lemma 3 and part two of this claim is positive. Furthermore f_{CA} is decreasing on I_9. Using inequality $\max_x f(x) + \max_x g(x) \geq \max_x f(x) + g(x)$ we can derive upper bound on $U_{CA}^{I_9}$ as we know the maxima of the $f_{CA}^{I_9}$ and $V_C^{I_9}$ functions.

The upper bound is given by

$$f_{CA}(I_9^L) + \delta V_C^{I_9}(I_9^U) \geq \max_{x \in I_9} U_{CA}^{I_9}(x)$$

and we need to show it is lower than $U_{CA}(\pi^* - \phi\delta p_d)$. Some algebra gives

$$1 - 3\delta p_d + 3\delta^2 p_d^2 + \frac{\delta^3 p_d^3}{1-\delta} \geq 0$$

which holds. To see this, we can disregard the last term in the expression which is positive. Regarding the remaining as a quadratic equation in δ gives pair of roots both of which are complex and it is easy to confirm the expression has to be positive.

The fifth part is indeed a crux of the proof as U_{CD} may have local maxima on I_9. First note that if we prove $U_{CD}(z(x)) \geq U_{CD}(x)$ $\forall x \in I_9$ then we are done by the fact the U_{CD} is decreasing on I_4 and $z(x) \in I_4$ $\forall x \in I_9$.

To start, we note the relevant parts of the V_C function can be alternatively expressed as

$$\begin{aligned}
V_C^{I_9}(x) =& p_d[f_{CD}(z(x)) + \delta V_C^{I_4}(z(x))] \\
&+ (1-p_d)[f_{CA}(\pi^* - \phi\delta p_d) + \delta V_C^{I_3}(\pi^* - \phi\delta p_d)] \\
V_C^{I_4}(x) =& p_d[f_{CD}(x) + \delta V_C^{I_4}(x)] \\
&+ (1-p_d)[f_{CA}(2(\pi^* + \phi\delta p_d) - x) + \delta V_C^{I_3}(2(\pi^* + \phi\delta p_d) - x)]
\end{aligned}$$

which upon substitution into $U_{CD}(z(x)) - U_{CD}(x)$ greatly simplifies the algebra as the first square brackets disappear. Nevertheless, some lengthy and uninstructive algebra finally gives

$$\begin{aligned}
&U_{CD}(z(x)) - U_{CD}(x) = \\
&\qquad 4\phi \left[(x - \pi^*) - \frac{1 - \delta - \delta^2 p_d + \delta^2 p_d^2}{1-\delta}(z(x) - \pi^*) - \frac{3\phi\delta^3 p_d^2 (1-p_d)}{1-\delta} \right].
\end{aligned}$$

It is easy to confirm this expression is positive for $x = I_9^L$. Taking the derivative with respect to x then gives

$$[U_{CD}(z(x)) - U_{CD}(x)]' = 4\phi\left[1 - \frac{1-\delta-\delta^2 p_d + \delta^2 p_d^2}{1-\delta} z(x)'\right]$$

which is positive. To see this notice $1 - \delta - \delta^2 p_d + \delta^2 p_d^2 > 0$ for $\delta \leq 1/(1+2p_d)$ and $z(x)'$ is negative by lemma 3. This proves the fifth part. Sixth part is then direct consequence of the above. $\square$

It is now easy to confirm the specified offers are indeed an equilibrium and can be written in a way used in the proposition 1.

By claim 4, CA either implements her global maximum $\pi^* - \phi\delta p_d$ or minimum of $A_A(x)$. This follows from the shape of A_A given in claim 2 which implies that if $\pi^* - \phi\delta p_d \notin A_A(x)$ for some x then $A_A(x) \in I_3 \cup I_4$.

For CD, the best option is when the global maximum $\pi^* - \phi$ is available. If she cannot implement her global optimum, then the lowest possible policy is implemented. This follows directly from claim 4 where the only problematic interval is I_9. But in claim 3 we have shown that for $x \in I_4$ the acceptance set takes the form $\langle x, z^{-1}(x)\rangle$ and for $x \in I_9$ the acceptance set takes the form $\langle z(x), x\rangle$. But then by part five of claim 4, CD implements as low policy as possible. This concludes the proof of case 1.

Case 2: Equilibrium for $\delta \geq \frac{1}{1+2p_d}$ and $4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d) \leq 0$

For $\delta \leq \frac{1}{1+2p_d}$ and $4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d) \leq 0$ the equilibrium offers are

$$p_A(x) = \begin{cases} \pi^* - \phi\delta p_d & \text{for } x \in I_1 \cup I_2 \cup I_5 \cup I_6 \cup I_{9-} \cup I_{9+} \cup I_{10} \cup I_{11} \cup I_{12} \\ x & \text{for } x \in I_3 \\ 2(\pi^* + \phi\delta p_d) - x & \text{for } x \in I_4 \cup I_7 \end{cases}$$

$$p_D(x) = \begin{cases} \pi^* - \phi & \text{for } x \in I_1 \cup I_{12} \\ x & \text{for } x \in I_2 \cup I_3 \cup I_4 \cup I_5 \\ 2(\pi^* + \phi(1-\delta(1-p_d))) - x & \text{for } x \in I_7 \\ 2(\pi^* + \phi) - x & \text{for } x \in I_6 \cup I_{11} \\ z(x) & \text{for } x \in I_{9-} \cup I_{9+} \cup I_{10} \end{cases}$$

where

$$\begin{aligned}
I_1 &= \langle x^-, \pi^* - \phi\rangle & I_5 &= (\tau_1^-, \pi^* + \phi\rangle \\
I_2 &= \langle \pi^* - \phi, \pi^* - \phi\delta p_d\rangle & I_6 &= \langle \pi^* + \phi, \tau_1^+) \\
I_3 &= \langle \pi^* - \phi\delta p_d, \pi^* + \phi\delta p_d\rangle & I_{9+} &= \langle \tau_1^+, \tau^+\rangle \\
I_4 &= \langle \pi^* + \phi\delta p_d, \pi^* + \phi(1-\delta(1-p_d))\rangle & I_{10} &= \langle \tau^+, \pi^* + \phi(2+\delta p_d)\rangle \\
I_7 &= \langle \pi^* + \phi(1-\delta(1-p_d)), \pi^* + 3\phi\delta p_d\rangle & I_{11} &= \langle \pi^* + \phi(2+\delta p_d), \pi^* + 3\phi\rangle \\
I_{9-} &= \langle \pi^* + 3\phi\delta p_d, \tau_1^-\rangle & I_{12} &= \langle \pi^* + 3\phi, x^+\rangle
\end{aligned}$$

where as before $\tau^+ = \pi^* + \phi + \phi\sqrt{(1-\delta p_d)^2 - \frac{4\delta^3 p_d^2(1-p_d)}{1-\delta}}$ and $\tau_1^\pm$ are defined as $\tau_1^- = \pi^* + \phi - \phi\sqrt{\frac{\delta(1-p_d)}{1-\delta}((1-\delta)(1-\delta p_d) - 4\delta^2 p_d^2)}$ and τ_1^+ analogously with the term involving the square root being added.

By condition on this case, the term under the square root in $\tau_1^\pm$ is positive. To see the term in the square root of τ^+ is positive, one follows the same procedure as for case 1 but instead of substituting $\delta = 1/(1+2p_d)$ one substitutes condition $\delta = 1/(1+p_d)$ which is indeed weaker condition than condition defining case 2, $4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d) \leq 0$.

It is matter of simple algebra to confirm that the equilibrium offers induce the continuation value functions specified above where I_{9+} and I_{9-} correspond to I_9. For V_P it is easy to show that the function is continuous everywhere and differentiable everywhere except at the borders of the intervals. For V_C it can be shown that it is differentiable everywhere except at the borders of the intervals. Regarding continuity, V_C is continuous everywhere except at I_5^L and I_6^U where it jumps in a discrete manner. This is a direct consequence of the equilibrium offers not being continuous at the same points with respect to the default policy x. We first pin down the shape of U_{PA} and U_{PD}.

claim 5 (Shape of U_{PA} and U_{PD})**.** *U_{PA} is increasing on $I_1 \cup I_2 \cup I_3$ and decreasing otherwise. U_{PD} is increasing on $I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5 \cup I_{9-}$ and decreasing otherwise. U_{PA} has global maximum at $\pi^* + \phi\delta p_d$ and is quasi-concave. U_{PD} has two local maxima at $\pi^* + \phi(1-\delta(1-p_d))$ and $\pi^* + \phi$ the latter of which is also a global maximum. U_{PD} has one local minimum at $\pi^* + 3\phi\delta p_d$.*

Proof. It is easy to show U_{PA} is increasing (and hence U_{PD} as well by lemma 1) on $I_1 \cup I_2 \cup I_3$. Similarly U_{PD} is decreasing (and hence U_{PA} by the same lemma) on $I_7 \cup I_6 \cup I_{9+} \cup I_{10} \cup I_{11} \cup I_{12}$. The remaining three intervals, I_4, I_{9-} and I_5, are equally easy. It follows U_{PA} has global maximum at $\pi^* + \phi\delta p_d$ which is border of I_3 with I_4 and its quasi-concavity follows. Similarly, U_{PD} has two local maxima. One at the border of I_4 and I_7 and the second at the border of I_5 and I_6. Also, it follows local minimum has to be at the border of I_7 and I_{9-}. It is easy to show $\pi^* + \phi$ is the global maximum. □

Next we wish to characterize the acceptance sets. As the shape of the A_A is exactly the same as in claim 2 we do not repeat it here. For the A_D we have the following.

claim 6 (Shape of $A_D(x)$)**.** *Let x be the default policy. Then*

1. *if $x \in I_1 \cup I_{12}$ then $\pi^* - \phi \in A_D(x)$*
2. *if $x \in I_2$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = 2(\pi^* + \phi) - x \in I_{11}$*
3. *if $x \in I_3 \cup \langle I_4^L, \pi^* + 2\phi(1-\delta(1+p_d/2))\rangle$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = z^{-1}(x) \in I_9 \cup I_{10}$*
4. *if $x \in \langle \pi^* + 2\phi(1-\delta(1+p_d/2)), I_4^U\rangle$ then $A_D(x) = A_D^1(x) \cup A_D^2(x)$ where $A_D^1 = \{p : x \leq p \wedge p \leq x'\}$, $A_D^2 = \{p : x'' \leq p \wedge p \leq x'''\}$, $x + x' =$*

$2(\pi^* + \phi(1-\delta(1-p_d))$, $x'' + x''' = 2(\pi^* + \phi)$, $x = z(x'') = z(x''')$, $x' \in I_7$, $x'' \in I_{9-}$ *and* $x''' \in I_{9+}$

5. *if* $x \in I_7$ *then* $A_D(x) = A_D^1(x) \cup A_D^2(x)$ *where* $A_D^1 = \{p : x' \leq p \wedge p \leq x\}$, $A_D^2 = \{p : x'' \leq p \wedge p \leq x'''\}$, $x + x' = 2(\pi^* + \phi(1-\delta(1-p_d))$, $x'' + x''' = 2(\pi^* + \phi)$, $x' = z(x'') = z(x''')$, $x' \in I_4$, $x'' \in I_{9-}$ *and* $x''' \in I_{9+}$

6. *if* $x \in I_{9-}$ *then* $A_D(x) = A_D^1(x) \cup A_D^2(x)$ *where* $A_D^1 = \{p : x'' \leq p \wedge p \leq x'''\}$, $A_D^2 = \{p : x \leq p \wedge p \leq x'\}$, $x'' + x''' = 2(\pi^* + \phi(1-\delta(1-p_d))$, $x + x' = 2(\pi^* + \phi)$, $x'' = z(x) = z(x')$, $x'' \in I_4$, $x''' \in I_7$ *and* $x' \in I_{9+}$

7. *if* $x \in \langle I_{9+}^L, \pi^* + \phi(2-3\delta p_d)\rangle$ *then* $A_D(x) = A_D^1(x) \cup A_D^2(x)$ *where* $A_D^1 = \{p : x'' \leq p \wedge p \leq x'''\}$, $A_D^2 = \{p : x' \leq p \wedge p \leq x\}$, $x'' + x''' = 2(\pi^* + \phi(1-\delta(1-p_d))$, $x + x' = 2(\pi^* + \phi)$, $x'' = z(x) = z(x')$, $x'' \in I_4$, $x''' \in I_7$ *and* $x' \in I_{9-}$

8. *if* $x \in I_5$ *then* $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ *where* $x' = 2(\pi^* + \phi) - x \in I_6$

9. *if* $x \in I_6$ *then* $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ *where* $x' = 2(\pi^* + \phi) - x \in I_5$

10. *if* $x \in \langle \pi^* + \phi(2-3\delta p_d), I_{9+}^U\rangle \cup I_{10}$ *then* $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ *where* $x' = z(x) \in I_3 \cup I_4$

11. *if* $x \in I_{11}$ *then* $A_D(x) = \{x' \leq p \wedge p \leq x\}$ *where* $x' = 2(\pi^* + \phi) - x \in I_2$.

Proof. Parts one through three and eight through eleven are very similar to the relevant parts in the claim 3. What we cannot use is the quasi-concavity of U_{PD}. However, it is easy to confirm that the acceptance sets are convex.

Parts four through seven present the key difference compared to claim 3. To see those, first notice for default policies specified, the U_{PD} is two-hill shaped. One of the hills is symmetric around $\pi^* + \phi(1-\delta(1-p_d))$ and the second one around $\pi^* + \phi$. It then follows $U_{PD}(x) = U_{PD}(x')$ gives four solutions. One pair symmetric around $\pi^* + \phi(1-\delta(1-p_d))$ and the second pair symmetric around $\pi^* + \phi$. It is then matter of straightforward algebra to work out the appropriate intervals. □

Following claim gives the shape of U_{CA} and U_{CD} functions.

claim 7 (Shape of U_{CA} and U_{CD}).

1. U_{CA} *is increasing on* $I_1 \cup I_2$ *and decreasing on* $I_3 \cup I_{9-} \cup I_5 \cup I_6 \cup I_{10} \cup I_{11} \cup I_{12}$

2. U_{CD} *is increasing on* I_1 *and decreasing on* $I_2 \cup I_3 \cup I_4 \cup I_7 \cup I_{9-} \cup I_5 \cup I_6 \cup I_{10} \cup I_{11} \cup I_{12}$

3. $U_{CA}(x) \geq U_{CA}(x')$ *where* $x \in I_3$ *and* $x' = 2(\pi^* + \phi\delta p_d) - x \in I_4 \cup I_7$

4. $U_{CA}(x'') \geq U_{CA}(x')$ *and* $U_{CD}(x'') \geq U_{CD}(x')$ *for every* $x' \in \langle I_{9+}^L, x\rangle$ *given* $x \in \langle I_{9+}^L, \pi^* + \phi(2-3\delta p_d)\rangle$ *with* $x'' = 2(\pi^* + \phi) - x \in I_{9-}$.

5. $U_{CA}(\pi^* - \phi\delta p_d) \geq \max_{x \in \langle \pi^* + \phi(2-3\delta p_d), I_{9+}^U\rangle} U_{CA}(x)$

6. $U_{CD}(z(x)) \geq U_{CD}(x')\ \forall x' \in \langle \pi^*+\phi(2-3\delta p_d), x\rangle$ *given* $x \in \langle \pi^* + \phi(2-3\delta p_d), I_{9+}^U\rangle$

7. U_{CA} *has global maximum at* $\pi^* - \phi\delta p_d$ *and* U_{CD} *at* $\pi^* - \phi$.

Proof. The first part is straightforward given the continuation value functions except for I_{10}. As in claim 4 we have V_C concave on this interval so if we can establish that $U_{CA}^{\prime I_{10}}(I_{10}^L)$ is negative the claim follows. In claim 4 this gave us equation

$$U_{CA}^{\prime I_{10}}(\tau^+) = -2\phi\left[1 + \left(\frac{\tau^+ - \pi^* - \phi}{\phi}\right)\left(\frac{1-\delta-2\delta p_d(1-\delta(1-p_d))}{(1-\delta)(1-\delta p_d)}\right)\right]$$

where we could establish negativity by the fact that $1-\delta-2\delta p_d(1-\delta(1-p_d)) > 0$. For the current case we need to do more work as this inequality might not be satisfied.

Note that $\frac{\tau^+-\pi^*-\phi}{\phi} < 1+\delta p_d$ which can be seen consulting the definition of the intervals I_i. Hence if we can prove the derivative is negative when $\frac{\tau^+-\pi^*-\phi}{\phi}$ is replaced by $1+\delta p_d$ the claim follows. Doing that gives

$$U_{CA}^{\prime I_{10}}(\tau^+) = -4\phi\left[\frac{1-\delta-\delta p_d(1-\delta(1-p_d))(1+\delta p_d)}{(1-\delta)(1-\delta p_d)}\right]$$

which is negative as the term in the square brackets is positive. To see that, take the nominator and substitute $\delta = (1+p_d-\sqrt{1-2p_d+17p_d^2})/(2p_d(1-4p_d))$ which is the solution to the condition defining case 2 and confirm the expression is positive. Next, taking the derivative of the nominator with respect to δ gives a quadratic equation in δ with the derivative being negative between the roots. One of the roots is negative and the second one is higher than unity. This shows the $U_{CA}^{\prime I_{10}}(\tau^+)$ is negative and hence proves the first part of the claim.

Second part of the claim is straightforward using the similar argument as part two of the claim 4. Likewise, the third part can be established using the same argument as part three of the claim 4 noting that width of I_3 is the same as width of $I_4 \cup I_7$.

To see the fourth part, notice that if we show $U_{CA}(x') \geq U_{CA}(x)$ and $U_{CD}(x') \geq U_{CD}(x)$ where $x' = 2(\pi^*+\phi) - x \in I_{9-}$ for every default policy $x \in \langle I_{9+}^L, \pi^* + \phi(2-3\delta p_d)\rangle$ then we are done. However, it is easy to confirm $V_C(x') = V_C(x)$ for x, x' just defined. Hence the claim follows.

The fifth part can be established using the similar argument as in part 4 of the claim 4 where the derivation of the upper bound on U_{CA}^{9+} is done using exactly the same values.

To prove the sixth part again the same argument as in part five of the claim 4 can be used. However, the conditions on δ defining case 2 alone are not sufficient to ensure $1-\delta-\delta^2 p_d + \delta^2 p_d^2 > 0$. However, the inequality still holds by the virtue of assumption 1. Finally, the last part is a direct consequence of the above. □

Again, putting claims 2, 6 and 7 together proves the specified offers are indeed an equilibrium. CA can either implement his overall optimum $\pi^* - \phi\delta p_d$ and when this policy is not available, she offers as low policy as possible.

The same logic applies for CD. Using the claim 7, CD either offers her overall optimum $\pi^* - \phi$ and if this is not available she offer as low policy as possible. This can be seen from the fact that U_{CD} is decreasing over majority of I_i intervals for policies above $\pi^* - \phi$. When we cannot establish the decreasing U_{CD}, claims 7 and 6 imply that whenever any policy from such interval is available, there is also available another policy that gives CD higher utility which in turn is rejected in favour of the lowest policy available. This concludes the proof of case 2.

Case 3: Equilibrium for $4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d) \geq 0$ and $\delta \leq \frac{1}{3p_d}$

For $4\delta^2 p_d^2 - (1-\delta)(1-\delta p_d) \geq 0$ and $\delta \leq \frac{1}{3p_d}$ the equilibrium offers are

$$p_A(x) = \begin{cases} \pi^* - \phi\delta p_d & \text{for } \; x \in I_1 \cup I_2 \cup I_{10-} \cup I_{9-} \cup I_{9+} \cup I_{10+} \cup I_{11} \cup I_{12} \\ x & \text{for } \; x \in I_3 \\ 2(\pi^* + \phi\delta p_d) - x & \text{for } \; x \in I_4 \cup I_7 \cup I_8 \end{cases}$$

$$p_D(x) = \begin{cases} \pi^* - \phi & \text{for } \; x \in I_1 \cup I_{12} \\ x & \text{for } \; x \in I_2 \cup I_3 \cup I_4 \\ 2(\pi^* + \phi(1 - \delta(1 - p_d))) - x & \text{for } \; x \in I_7 \cup I_8 \\ z(x) & \text{for } \; x \in I_{10-} \cup I_{9-} \cup I_{9+} \cup I_{10} \\ 2(\pi^* + \phi) - x & \text{for } \; x \in I_{11} \end{cases}$$

where

$$\begin{aligned} I_1 &= \langle x^-, \pi^* - \phi \rangle \\ I_2 &= \langle \pi^* - \phi, \pi^* - \phi\delta p_d \rangle \\ I_3 &= \langle \pi^* - \phi\delta p_d, \pi^* + \phi\delta p_d \rangle \\ I_4 &= \langle \pi^* + \phi\delta p_d, \pi^* + \phi(1 - \delta(1 - p_d)) \rangle \\ I_7 &= \langle \pi^* + \phi(1 - \delta(1 - p_d)), \pi^* + 2\phi(1 - \delta(1 - p_d/2)) \rangle \\ I_8 &= \langle \pi^* + 2\phi(1 - \delta(1 - p_d/2)), \pi^* + 3\phi\delta p_d \rangle \end{aligned}$$

$$\begin{aligned} I_{10-} &= \langle \pi^* + 3\phi\delta p_d, \tau^- \rangle \\ I_{9-} &= \langle \tau^-, \pi^* + \phi \rangle \\ I_{9+} &= \langle \pi^* + \phi, \tau^+ \rangle \\ I_{10+} &= \langle \tau^+, \pi^* + \phi(2 + \delta p_d) \rangle \\ I_{11} &= \langle \pi^* + \phi(2 + \delta p_d), \pi^* + 3\phi \rangle \\ I_{12} &= \langle \pi^* + 3\phi, x^+ \rangle. \end{aligned}$$

Case 3 indeed subsumes two important subcases depending on whether $\delta \leq 1/(1 + p_d)$ holds and one of the subcases can even be split further. However, to economize on space and avoid extensive repetition of similar arguments we have decided to treat all the subcases at once.

One should then be aware that some of the I_i intervals above might not be properly defined. For $\delta \geq 1/(1+p_d)$ the intervals are exactly as those just given with the qualification that I_{9-} and I_{9+} might not exist if τ^- and τ^+ become complex. If this happens, then I_{10-} and I_{10+} naturally extend all the way to $\pi^* + \phi$. If below we need to distinguish those two cases, we refer to case 3.1 if $\delta \geq 1/(1 + p_d)$ with $\tau^{\pm}$ real and to case 3.2 if $\delta \geq 1/(1 + p_d)$ with $\tau^{\pm}$ complex.

The remaining possibility, referred to as case 3.3, is when $\delta \leq 1/(1 + p_d)$ in which case I_8 ceases to exist and I_7 extends all the way to $\pi^* + 3\phi\delta p_d$. If this happens, I_{10-} also ceases to exist and I_{9-} starts immediately at $\pi^* + 3\phi\delta p_d$.

As before, the equilibrium offers induce the continuation value functions given above where I_{9-} and I_{9+} map into I_9 and analogously for $I_{10\pm}$. Both V_C and V_P are continuous everywhere and differentiable everywhere except at the borders of I_i intervals. Proceeding similarly, we first describe the shape of U_{PA} and U_{PD}.

claim 8 (Shape of U_{PA} and U_{PD})**.** *U_{PA} is increasing on $I_1 \cup I_2 \cup I_3$ and decreasing otherwise. U_{PD} is increasing on $I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_{10-} \cup I_{9-}$ and decreasing otherwise. U_{PA} has global maximum at $\pi^* + \phi\delta p_d$ and is quasi-concave. U_{PD} has two local maxima at $\pi^* + \phi(1 - \delta(1 - p_d))$ and $\pi^* + \phi$ the former of which is also a global maximum. U_{PD} has one local minimum at $\pi^* + 3\phi\delta p_d$.*

Proof. The argument is essentially as in claim 5 adjusting for different intervals. The key difference is that the global maximum is at $\pi^* + \phi(1 - \delta(1 - p_d))$ and not at $\pi^* + \phi$, something that can be readily verified. □

To characterize the shape of the acceptance sets, the A_A described in claim 2 applies for the current case as well and we do not repeat it here. Before we pin down A_D let us define another pair of constants $\tau_2^{\pm}$ given by the expression $\tau_2^- = \pi^* + \phi(1 - \delta(1 - p_d)) - \phi\sqrt{\frac{\delta(1-p_d)}{1-\delta p_d}(4\delta^2 p_d^2 - (1 - \delta)(1 - \delta p_d))}$ and analogously for τ_2^+. Notice that by one of the conditions defining case 3, the term in the square root is positive. With this definition we have the following.

claim 9 (Shape of $A_D(x)$)**.** *Let x be the default policy. Then*

1. *if $x \in I_1 \cup I_{12}$ then $\pi^* - \phi \in A_D(x)$*
2. *if $x \in I_2$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = 2(\pi^* + \phi) - x \in I_{11}$*
3. *if $x \in \langle I_3^L, \pi^* + 2\phi(1 - \delta(1 + p_d/2))\rangle$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = z^{-1}(x) \in I_{9+} \cup I_{10+}$*
4. *if $x \in \langle \pi^* + 2\phi(1 - \delta(1 + p_d/2)), \tau_2^-\rangle$ then $A_D(x) = A_D^1(x) \cup A_D^2(x)$ where $A_D^1 = \{p : x \leq p \wedge p \leq x'\}$, $A_D^2 = \{p : x'' \leq p \wedge p \leq x'''\}$, $x + x' = 2(\pi^* + \phi(1 - \delta(1 - p_d))$, $x'' + x''' = 2(\pi^* + \phi)$, $x = z(x'') = z(x''')$, $x' \in I_7 \cup I_8$, $x'' \in I_{10-} \cup I_{9-}$ and $x''' \in I_{9+} \cup I_{10+}$*
5. *if $x \in \langle \tau_2^+, \pi^* + 3\phi\delta p_d\rangle$ then $A_D(x) = A_D^1(x) \cup A_D^2(x)$ where $A_D^1 = \{p : x' \leq p \wedge p \leq x\}$, $A_D^2 = \{p : x'' \leq p \wedge p \leq x'''\}$, $x + x' = 2(\pi^* + \phi(1 - \delta(1 - p_d))$, $x'' + x''' = 2(\pi^* + \phi)$, $x' = z(x'') = z(x''')$, $x' \in I_3 \cup I_4$, $x'' \in I_{10-} \cup I_{9-}$ and $x''' \in I_{9+} \cup I_{10+}$*
6. *if $x \in I_{10-} \cup I_{9-}$ then $A_D(x) = A_D^1(x) \cup A_D^2(x)$ where $A_D^1 = \{p : x'' \leq p \wedge p \leq x'''\}$, $A_D^2 = \{p : x \leq p \wedge p \leq x'\}$, $x'' + x''' = 2(\pi^* + \phi(1 - \delta(1 - p_d))$, $x + x' = 2(\pi^* + \phi)$, $x'' = z(x) = z(x')$, $x'' \in I_3 \cup I_4$, $x''' \in I_7 \cup I_8$ and $x' \in I_{9+} \cup I_{10+}$*

7. *if* $x \in \langle I^L_{9+}, \pi^* + \phi(2 - 3\delta p_d)\rangle$ *then* $A_D(x) = A^1_D(x) \cup A^2_D(x)$ *where* $A^1_D = \{p : x'' \leq p \wedge p \leq x'''\}$, $A^2_D = \{p : x' \leq p \wedge p \leq x\}$, $x'' + x''' = 2(\pi^* + \phi(1 - \delta(1 - p_d))$, $x + x' = 2(\pi^* + \phi)$, $x'' = z(x) = z(x')$, $x'' \in I_3 \cup I_4$, $x''' \in I_7 \cup I_8$ *and* $x' \in I_{10-} \cup I_{9-}$

8. *if* $x \in \langle \tau_2^-, \pi^* + \phi(1 - \delta(1 - p_d))\rangle$ *then* $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ *where* $x' = 2(\pi^* + \phi(1 - \delta(1 - p_d))) - x \in I_7 \cup I_8$

9. *if* $x \in \langle \pi^* + \phi(1 - \delta(1 - p_d)), \tau_2^+\rangle$ *then* $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ *where* $x' = 2(\pi^* + \phi(1 - \delta(1 - p_d))) - x \in I_3 \cup I_4$

10. *if* $x \in \langle \pi^* + \phi(2 - 3\delta p_d), I^U_{10+}\rangle$ *then* $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ *where* $x' = z(x) \in I_3 \cup I_4$

11. *if* $x \in I_{11}$ *then* $A_D(x) = \{x' \leq p \wedge p \leq x\}$ *where* $x' = 2(\pi^* + \phi) - x \in I_2$.

Proof. The proof is very similar to the proof of claim 6 where the key difference arises due to the fact that the higher of the two hills is the one symmetric around $\pi^* + \phi(1 - \delta(1 - p_d))$. □

To finish the proof of the case 3, we need to show C indeed wants to implement as low policy as possible. Next claim proves that.

claim 10 (Shape of U_{CA} and U_{CD})**.**

1. U_{CA} *is increasing on* $I_1 \cup I_2$ *and decreasing on* $I_3 \cup I_{10-} \cup I_{9-} \cup I_{11} \cup I_{12}$

2. U_{CD} *is increasing on* I_1 *and decreasing on* $I_2 \cup I_3 \cup I_4 \cup I_{10-} \cup I_{9-} \cup I_{11} \cup I_{12}$

3. $U_{CA}(x) \geq U_{CA}(x')$ *where* $x \in I_3$ *and* $x' = 2(\pi^* + \phi\delta p_d) - x \in I_4 \cup I_7 \cup I_8$

4. $U_{CD}(x) \geq U_{CD}(x')$ *where* $x \in I_3 \cup I_4$ *and* $x' = 2(\pi^* + \phi(1 - \delta(1 - p_d))) - x \in I_7 \cup I_8$

5. $U_{CA}(x'') \geq U_{CA}(x')$ *and* $U_{CD}(x'') \geq U_{CD}(x')$ *for every* $x' \in \langle I^L_{9+}, x\rangle$ *given* $x \in \langle I^L_{9+}, \pi^* + \phi(2 - 3\delta p_d)\rangle$ *with* $x'' = 2(\pi^* + \phi) - x \in I_{10-} \cup I_{9-}$.

6. U_{CA} *and* U_{CD} *are decreasing on* $\langle \pi^* + \phi(2 - 3\delta p_d), I^U_{10+}\rangle$

7. U_{CA} *has global maximum at* $\pi^* - \phi\delta p_d$ *and* U_{CD} *at* $\pi^* - \phi$.

Proof. The first and second part of the claim can be readily verified using the expressions for the continuation values.

Part three can be established using the lemma 3 where we note that we are allowed to use it given that the width of I_3 is the same as width of $I_4 \cup I_7 \cup I_8$. The same argument gives part four as the width of $I_3 \cup I_4$ is larger that the width of $I_7 \cup I_8$.

To see the fifth part, notice that if we show that $U_{CA}(x') \geq U_{CA}(x)$ and $U_{CD}(x') \geq U_{CD}(x)$ with $x' = 2(\pi^* + \phi) - x \in I_{10-} \cup I_{9-}$ for every default policy $x \in \langle I^L_{9+}, \pi^* + \phi(2 - 3\delta p_d)$ then we are done. However, it is easy to confirm $V_C(x') = V_C(x)$ for x, x' just defined and the claim follows.

Part six is the key difficulty. Note that by lemma 1 it suffices to show U_{CA} decreasing. However, we cannot rely on concavity of V_C as in claims 4 and 7. Instead we will use a following strategy. Writing $U'_{CA}(x) = f'_{CA}(x) + \delta V'_C(x)$ we replace $V'_C(x)$ by upper bound on its maximum on appropriate interval and show the resulting expression is negative which also proves that U_{CA} is decreasing.

Here we are forced to split the proof according to different cases. For cases 3.1 and 3.2 the interval $\langle \pi^* + \phi(2-3\delta p_d), I^U_{10+}\rangle$ falls into I_{10+} and we can write

$$V_C'^{I_{10+}}(x) = p_d z(x)' \left[U_{CD}'^{I_3}(z(x)) \right]$$

where we want to find upper bound on maximum of $V_C'^{I_{10+}}$ on the interval $\langle \pi^* + \phi(2-3\delta p_d), I^U_{10+}\rangle$. To do so notice both of the terms are negative and hence if we can find minima of the two terms treated separately this will give us something that has to be higher than the maximum of $V_C'^{I_{10+}}$.

It is easy to establish $z(x)'$ is decreasing on I_{10+} while the term in the square brackets is increasing on I_{10+}. It follows that if we evaluate $z(x)'$ at I^U_{10+} and $U_{CD}'^{I_3}(z(x))$ at $\pi^* + \phi(2-3\delta p_d)$ the resulting expression will give us upper bound on the maximum of $V'_C(x)$ on $\langle \pi^* + \phi(2-3\delta p_d), I^U_{10+}\rangle$. Doing so gives

$$\min_{x \in \langle \pi^* + \phi(2-3\delta p_d), I^U_{10+}\rangle} z(x)' \geq -1$$

$$\min_{x \in \langle \pi^* + \phi(2-3\delta p_d), I^U_{10+}\rangle} U'_{CD}(z(x)) = -6\phi$$

which gives us maximum for V'_C. It is then matter of straightforward algebra to substitute the maximum into $U'_{CA}(x) = f'_{CA}(x) + \delta V'_C(x)$ and confirm the resulting expression is negative on $\langle \pi^* + \phi(2-3\delta p_d), I^U_{10+}\rangle$.

For case 3.3, $\pi^* + \phi(2-3\delta p_d) \in I_{9+}$ so that we need to use similar argument but separately on $\langle \pi^* + \phi(2-3\delta p_d), I^U_{9+}\rangle$ and I_{10+}. We can still use

$$V_C'^{I_{9+}}(x) = p_d z(x)' \left[U_{CD}'^{I_4}(z(x)) \right] \qquad V_C'^{I_{10+}}(x) = p_d z(x)' \left[U_{CD}'^{I_3}(z(x)) \right]$$

and the fact that $z(x)'$ is decreasing on $I_{9+} \cup I_{10+}$ and $U_{CD}'^{I_4}(z(x))$ with $U_{CD}'^{I_3}(z(x))$ are increasing on I_{9+} and I_{10+} respectively. It follows we need to evaluate $z(x)'$ at I^U_{9+} and I^U_{10+}, $U_{CD}'^{I_4}(z(x))$ at $\pi^* + \phi(2-3\delta p_d)$ and $U_{CD}'^{I_3}(z(x))$ at I^L_{10+}.

The evaluation gives

$$\min_{x \in \langle \pi^* + \phi(2-3\delta p_d), I^U_{9+}\rangle \cup I_{10+}} z(x)' \geq -1$$

$$\min_{x \in \langle \pi^* + \phi(2-3\delta p_d), I^U_{9+}\rangle} U'_{CD}(z(x)) = -\frac{2\phi}{(1-\delta)(1-\delta p_d)}(3(1-\delta-\delta p_d+\delta^2 p_d^2) - \delta^2 p_d(1-p_d))$$

$$\min_{x \in I_{10+}} U'_{CD}(z(x)) = -\frac{2\phi}{1-\delta}(1-\delta+2\delta p_d).$$

Upon substitution of the maximum of V'_C into $U'_{CA}(x) = f'_{CA}(x) + \delta V'_C(x)$ the condition for U_{CA} decreasing on I_{10+} becomes

$$\frac{\delta p_d}{1-\delta}(1-\delta+2\delta p_d) - 1 - \sqrt{(1-\delta p_d)^2 - \frac{4\delta^3 p_d^2(1-p_d)}{1-\delta}} \leq 0$$

which holds. To see this notice that for $p_d \leq 1/2$ we are done. Otherwise, substituting $\delta = 1/(1+p_d)$ confirms the condition holds for maximum δ allowed for the case 3.3. Then the derivative of the condition with respect to δ is positive and hence the condition must hold. Therefore U_{CA} (and hence U_{CD} by lemma 1) is decreasing on I_{10+}.

For $\langle \pi^* + \phi(2 - 3\delta p_d), I^U_{9+} \rangle$ upon substitution the corresponding condition is $(1-\delta)(4\delta p_d - 1) - 3\delta^2 p_d^2(1 - \delta p_d) + \delta^3 p_d^2(1 - p_d) \leq 0$ which holds for the case 3.3. To see this regard it as a cubic equation in δ. Solving for the roots, noticing that the condition holds for δ below the lowest root and showing that the lowest root is higher than $1/3p_d$ proves the claim. Finally the last part of the claim follows from all the above. □

Combining the information provided by claims 2, 9 and 10 proves the equilibrium for case 3. CA either offers her overall optimum $\pi^* - \phi\delta p_d$ and when this policy is not available, then she offers as low policy as possible. This follows from the information about the intervals over which U_{CA} is decreasing provided by claim 10 and where we cannot use this argument the same claim implies that the minimum policy available gives CA highest utility among the policies available. The same argument applies for CD and concludes the proof for case 3.

Case 4: Equilibrium for $\delta \geq \frac{1}{3p_d}$

For $\delta \geq \frac{1}{3p_d}$ the equilibrium offers are

$$p_A(x) = \begin{cases} \pi^* - \phi\delta p_d & \text{for } x \in I_1 \cup I_2 \cup I_9 \cup I_{10} \cup I_{11} \cup I_{12} \\ x & \text{for } x \in I_3 \\ 2(\pi^* + \phi\delta p_d) - x & \text{for } x \in I_4 \cup I_7 \cup I_8 \end{cases}$$

$$p_D(x) = \begin{cases} \pi^* - \phi & \text{for } x \in I_1 \cup I_{12} \\ x & \text{for } x \in I_2 \cup I_3 \cup I_4 \\ 2(\pi^* + \phi(1 - \delta(1 - p_d))) - x & \text{for } x \in I_7 \cup I_8 \\ z(x) & \text{for } x \in I_9 \cup I_{10} \\ 2(\pi^* + \phi) - x & \text{for } x \in I_{11} \end{cases}$$

where

$$I_1 = \langle x^-, \pi^* - \phi \rangle$$
$$I_2 = \langle \pi^* - \phi, \pi^* - \phi\delta p_d \rangle$$
$$I_3 = \langle \pi^* - \phi\delta p_d, \pi^* + \phi\delta p_d \rangle$$
$$I_4 = \langle \pi^* + \phi\delta p_d, \pi^* + \phi(1 - \delta(1 - p_d)) \rangle$$
$$I_7 = \langle \pi^* + \phi(1 - \delta(1 - p_d)), \pi^* + 2\phi(1 - \delta(1 - p_d/2)) \rangle$$
$$I_8 = \langle \pi^* + 2\phi(1 - \delta(1 - p_d/2)), \pi^* + 3\phi\delta p_d \rangle$$
$$I_9 = \langle \pi^* + 3\phi\delta p_d, \tau^+ \rangle$$
$$I_{10} = \langle \tau^+, \pi^* + \phi(2 + \delta p_d) \rangle$$
$$I_{11} = \langle \pi^* + \phi(2 + \delta p_d), \pi^* + 3\phi \rangle$$
$$I_{12} = \langle \pi^* + 3\phi, x^+ \rangle.$$

As in the previous case we have subsumed two subcases and prove the equilibrium for those jointly. The first subcase, referred to as case 4.1, is for

$\delta \geq 1/(1+p_d)$. If this condition holds all the intervals are as those given except for I_9 which does not exist and I_{10} starts at $\pi^* + 3\phi\delta p_d$. For $\delta \leq 1/(1+p_d)$, referred to as case 4.2, the interval I_8 does not exist and I_7 extends all the way to $\pi^* + 3\phi\delta p_d$.

Once again it is easy to confirm the strategies given induce continuation value functions on the corresponding intervals. For the current case both V_C and V_P are continuous everywhere and differentiable everywhere except for points where the different I_i intervals meet. Proceeding similarly, we first give the properties of U_{PA} and U_{PD}.

claim 11 (Shape of U_{PA} and U_{PD})**.** *U_{PA} is increasing on $I_1 \cup I_2 \cup I_3$ and decreasing otherwise. U_{PD} is increasing on $I_1 \cup I_2 \cup I_3 \cup I_4$ and decreasing otherwise. U_{PA} has global maximum at $\pi^* + \phi\delta p_d$, U_{PD} has global maximum at $\pi^* + \phi(1-\delta(1-p_d))$ and both functions are quasi-concave.*

Proof. The argument is very similar to the one used to prove claim 1 with minor adjustments for the fact that U_{CD} has global maximum at $\pi^* + \phi(1-\delta(1-p_d))$ which is immediately apparent upon realizing that $\pi^* + \phi(1-\delta(1-p_d))$ is a border of I_4 and I_7. □

Proceeding to outline the shape of the acceptance sets, for A_A the claim 2 applies for the current case as well and we do not repeat it here. For A_D we have following.

claim 12 (Shape of $A_D(x)$)**.** *Let x be the default policy. Then*

1. *if $x \in I_1 \cup I_{12}$ then $\pi^* - \phi \in A_D(x)$*
2. *if $x \in I_2$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = 2(\pi^* + \phi) - x \in I_{11}$*
3. *if $x \in \langle I_3^L, \pi^* + 2\phi(1-\delta(1+p_d/2))\rangle$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = z^{-1}(x) \in I_9 \cup I_{10}$*
4. *if $x \in \langle \pi^* + 2\phi(1-\delta(1+p_d/2)), \pi^* + \phi(1-\delta(1-p_d))\rangle$ then $A_D(x) = \{p : x \leq p \wedge p \leq x'\}$ where $x' = 2(\pi^* + \phi(1-\delta(1-p_d))) - x \in I_7 \cup I_8$*
5. *if $x \in I_7 \cup I_8$ then $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ where $x' = 2(\pi^* + \phi(1-\delta(1-p_d))) - x \in I_3 \cup I_4$*
6. *if $x \in I_9 \cup I_{10}$ then $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ where $x' = z(x) \in I_3 \cup I_4$*
7. *if $x \in I_{11}$ then $A_D(x) = \{p : x' \leq p \wedge p \leq x\}$ where $x' = 2(\pi^* + \phi) - x \in I_2$.*

Proof. The proof is very similar to the proof of claim 3 where only minor adjustments have to be made for the current case due to the fact that U_{CD} is symmetric around its global maximum at $\pi^* + \phi(1-\delta(1-p_d))$ and hence some of the acceptance sets have to be made symmetric around $\pi^* + \phi(1-\delta(1-p_d))$. □

Having the acceptance sets the last thing we need to do is to pin down the shape of U_{CA} and U_{CD}. Next claim does that.

claim 13 (Shape of U_{CA} and U_{CD})**.**

1. *U_{CA} is increasing on $I_1 \cup I_2$ and decreasing on $I_3 \cup I_9 \cup I_{10} \cup I_{11} \cup I_{12}$*
2. *U_{CD} is increasing on I_1 and decreasing on $I_2 \cup I_3 \cup I_4 \cup I_9 \cup I_{10} \cup I_{11} \cup I_{12}$*
3. *$U_{CA}(x) \geq U_{CA}(x')$ where $x \in I_3$ and $x' = 2(\pi^* + \phi\delta p_d) - x \in I_4 \cup I_7 \cup I_8$*
4. *$U_{CD}(x) \geq U_{CD}(x')$ where $x \in I_3 \cup I_4$ and $x' = 2(\pi^* + \phi(1-\delta(1-p_d))) - x \in I_7 \cup I_8$*
5. *U_{CA} has global maximum at $\pi^* - \phi\delta p_d$ and U_{CD} at $\pi^* - \phi$.*

Proof. The first and second parts of the claim follow readily using the continuation value function, except for intervals I_9 and I_{10}.

For case 4.1 we do not have to worry about I_9 as it is empty. To show U_{CA} is decreasing on I_{10} we use the same argument as in claim 10. The only possible difference arises from the fact that in case 3.1 the relevant part of the claim 10 $U_{CD}'^{I_3}(z(x))$ has been evaluated at $\pi^* + \phi(2 - 3\delta p_d)$ whereas for the case 4.1 we need to evaluate $U_{CD}'^{I_3}(z(x))$ at $\pi^* + 3\phi\delta p_d$. However, it is easy to confirm that $U_{CD}'^{I_3}(z(\pi^* + 3\phi\delta p_d)) = U_{CD}'^{I_3}(z(\pi^* + \phi(2 - 3\delta p_d)))$ and the argument is essentially the same.

For case 4.2 we need to show the claim for both, I_9 as well as I_{10}. Nevertheless, the resulting expressions for maximum of V_C' on appropriate intervals are the same as in case 3.3 of the relevant part of claim 10. This is due to the fact that the only change is that I_9 starts at $\pi^* + 3\phi\delta p_d$ not at $\pi^* + \phi(2 - 3\delta p_d)$ but $z(x)$ evaluated at those values is the same. Therefore for the I_{10} interval the claim follows by the similar argument as in claim 10. For I_9 the condition for U_{CA} to be decreasing becomes (note this change is due to the fact that the I_9^L now is different than in claim 10) $-\frac{4\delta^3 p_d^2(1-\delta)}{(1-\delta)(1-\delta p_d)} \leq 0$ which holds.

Finally the parts three and four follow by the use of lemma 3 where we note that we can use it as a width of I_3 is the same as $I_4 \cup I_7 \cup I_8$ (part three) and a width of $I_3 \cup I_4$ is larger than the width of $I_7 \cup I_8$ (part four). Part five then follows from the previous parts. □

By now familiar argument we do not repeat here we have an equilibrium for case 4.

Uniqueness

To prove the essential uniqueness of the equilibrium, we first establish properties of the acceptance correspondences A_D and A_A.

claim 14. *For any $x \in X$ the acceptance correspondences $A_D(x)$ and $A_A(x)$ are nonempty, compact valued and upper-hemicontinuous.*

Proof. The nonempty and compact valued parts of the claim follow by definition. To prove upper-hemicontinuity of the acceptance correspondence

$$A_D(x) = \{p \in X | U_{PD}(p) \geq U_{PD}(x)\}$$

pick two sequences $\{x_\alpha\} \to x$ and $\{p_\alpha\} \to p$ such that $p_\alpha \in A_D(x_\alpha)\ \forall \alpha$. Note that by non-emptiness of A_D this can be done. We need to show $p \in A_D(x)$.

Suppose $p \notin A_D(x)$. Then

$$\begin{aligned} U_{PD}(x_\alpha) &\leq U_{PD}(p_\alpha)\ \forall \alpha \\ U_{PD}(x) &> U_{PD}(p). \end{aligned}$$

Summing the two inequalities gives

$$U_{PD}(x_\alpha) - U_{PD}(x) < U_{PD}(p_\alpha) - U_{PD}(p)\ \forall \alpha.$$

Taking the limit for $\alpha \to \infty$ on both sides gives contradiction to continuity of $U_{PD}(\cdot)$. For A_A the proof is analogous and hence omitted. $\square$

We note that although we have proven upper-hemicontinuity of the acceptance correspondences, for some of the cases above we could prove continuity as well. More specifically, for all cases A_A can be proven continuous and for cases 1 and 4, A_D is continuous as well. Given that we do not need this stronger result, we state it without proving.

Another interesting question arises as to what is the reason for failure of lower-hemicontinuity of A_D in cases 2 and 3. As shown in claims 5 and 8 the shape of U_{PD} resembles two peaks. The lower of the two is the reason. We can always find a sequence of policies approaching the higher summit as A_D is nonempty. On the other hand there is no way to find a sequence of policies approaching the lower summit 'from above'.

Returning to our main argument, to prove the uniqueness result we need to show uniqueness of the solution of the system of functional Bellman equations

$$\begin{aligned} U_D(x) &= \max_{p \in A_D(x)} \{f_{CD}(p) + \delta p_d U_D(p) + \delta(1 - p_d) U_A(p)\} \\ U_A(x) &= \max_{p \in A_A(x)} \{f_{CA}(p) + \delta p_d U_D(p) + \delta(1 - p_d) U_A(p)\} \end{aligned}$$

where $V_C(x) = p_d U_D(x) + (1 - p_d) U_A(x)$. We already know the acceptance correspondences of the system are upper-hemicontinuous. If we could prove their continuity we would be able to use theorem 4.6 in Stokey and Lucas (1989) to prove the uniqueness of the solution to the system above. It turns out the result holds for upper-hemicontinuous correspondence as well, given we are willing to make a concession to the value functions being merely upper-semicontinuous and not continuous as in Stokey and Lucas (1989). The following theorem states the result formally.

Theorem 1. *Let X be convex subset of $\mathbb{R}^n$, $\Gamma : X \twoheadrightarrow X$ nonempty, compact valued and upper hemicontinuous corresponence, $F : A \to \mathbb{R}$ on $A = \{(x, y) \in$*

$X \times X|\ y \in \Gamma(x)\}$ bounded and upper semicontinuous function, $SC(X)$ space of bounded upper semicontinuous functions $f : X \to R$ with the sup norm $\|f\| = \sup_{x \in X} |f(x)|$ and $\beta < 1$. Then the T operator defined by

$$(Tf)(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta f(y)] \tag{4}$$

maps $SC(X)$ into itself and has a unique fixed point $v = Tv$.

Proof. Strategy of the proof is the following. First we make sure the maximum in (4) exists, next we show that T is upper semicontinuous (u.s.c.) and hence maps $SC(X)$ into itself. Next we observe T is a contraction and hence has unique fixed point, provided $SC(X)$ is complete. As is customary, we view normed vector space $(X, \|\cdot\|)$ as a metric space on X with the uniform metric $d(f, g) = \|f - g\|$.

Since the notion of upper semicontinuity is not well known in the economic literature we provide its definition.

Definition 4 (upper semicontinuous function)**.** *A function $f : X \to \bar{\mathbb{R}}$ on a topological space X is upper semicontinuous at $x \in X$ if for each $\epsilon > 0$ there exists a neighbourhood U of x such that $f(y) \leq f(x) + \epsilon$ for all y in U. It is upper semicontinuous if it is upper semicontinuous at $\forall x \in X$.*

An alternative definition sometimes used takes sequence $\{x_n\}$ and defines u.s.c. as a function that satisfies $x_n \to x \Rightarrow \limsup_n f(x_n) \leq f(x)$ which is indeed the same requirement (Bourbaki (2007), chapter IV.6, proposition 4). Yet another definition requires the $\{x \in X | f(x) < c\}$ to be open for any $c \in \mathbb{R}$ which is proved to be equal to the previous definition in Aliprantis and Border (2006), lemma 2.42.

Intuitively, u.s.c. functions are allowed to jump but when they do so, the value of the function at the jump is 'the higher of the two'. The advantage of the u.s.c. functions is that they posses a maximum on the compact interval.

Coming back to the actual proof, first observe that for any $x \in X$ the function $F(x, \cdot) + \beta f(\cdot)$ is u.s.c. and is being maximized on a compact, non-empty set $\Gamma(x)$, hence the maximum exists (Aliprantis and Border (2006), theorem 2.43).

Furthermore, as Γ is upper hemicontinuous, T is u.s.c. (Aliprantis and Border (2006), lemma 17.30) and it is clearly bounded. Hence $T : SC(X) \to SC(X)$.

Next we need to make sure T satisfies conditions under which Blackwell's Theorem (Aliprantis and Border (2006), theorem 3.53) holds. Denoting by $B(X)$ space of bounded functions defined on X, we need T to map closed linear subspace of $B(X)$ that includes constant functions into itself. Furthermore, we need T to satisfy *monotonicity* and *discounting*.

That $SC(X)$ is a linear subspace of $B(X)$ which includes constant functions follows trivially. To establish $SC(X)$ is closed we observe that $B(X)$ is complete and that any complete subset of a complete metric space is closed (Berberian

(1999), chapter III.4, theorem 1). Hence if we can establish that $SC(X)$ is complete closedness follows.

To establish $SC(X)$ with the uniform metric is a complete metric space, we adopt the approach of proof of theorem 3.1 in Stokey and Lucas (1989) with appropriate modifications. We find the function f to which Cauchy sequence of functions $\{f_n\}$ converges, we show the sequence converges in the uniform metric and finally that $f \in SC(X)$.

First, fix $x \in X$ and take a sequence $\{f_n(x)\}$ which satisfies

$$|f_n(x) - f_m(x)| \leq \sup_{y \in X} |f_n(y) - f_m(y)| = \|f_n - f_m\|$$

which satisfies the Cauchy criterion and hence converges to a limit $f(x)$.

Second, we need to show $\{f_n\}$ converges in the uniform metric. Pick $\epsilon > 0$ and $N := N(\epsilon)$ such than $n, m \geq M \Rightarrow \|f_n - f_m\| \leq \epsilon/2$ (which can be done). For any $x \in X$ and all $n, m \geq N$

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(m)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\| + |f_m(x) - f(x)| \\ &\leq \epsilon/2 + |f_m(x) - f(x)|. \end{aligned}$$

As $f_m(x) \to f(x)$, choose $m(x)$ for each $x \in X$ such that $|f_m(x) - f(x)| \leq \epsilon/2$. As x was arbitrary, it follows $\|f_n - f\| \leq \epsilon$ for $\forall n \geq N$ and as ϵ was arbitrary, we have convergence in the uniform metric.

Third, we need to show f is bounded and u.s.c. first of which follows readily. To show u.s.c., pick $\epsilon > 0$ and k such that $\|f_k - f\| \leq \epsilon/3$. As $f_n \to f$ this can be done. Then choose δ such that $\|x - y\|_E < \delta \Rightarrow f_k(y) < f_k(x) + \epsilon/3$ where $\|\cdot\|_E$ is usual Euclidean distance and it can be done by u.s.c. of f_k. Finally

$$\begin{aligned} f(y) - f(x) &= f(y) - f_k(y) + f_k(y) - f_k(x) + f_k(x) - f(x) \\ &\leq |f(y) - f_k(y)| + f_k(y) - f_k(x) + |f_k(x) - f(x)| \\ &\leq 2\|f - f_k\| + f_k(y) - f_k(x) \\ &\leq \epsilon. \end{aligned}$$

Finally, it is easy to confirm that $q \leq f$ implies $Tq \leq Tf$ (monotonicity) and that there exists $\beta \in (0, 1)$ such that $T(f + c) \leq Tf + \beta c$ for any constant function c (discounting). Hence by Blackwell's Theorem T is a contraction and has a unique fixed point which concludes the proof. □

It can be readily verified that we can use the theorem in the current setting. With the existence result in hand, it is obvious that the first equation has unique solution for each U_A and the second equation has unique solution for each U_D. It follows there exists a unique pair U_D^*, U_A^* that solves the system as a whole (in the mathematical literature on this topic this is called coincidence solution).

Now notice that we have started the derivation of the equilibrium with a conjecture that C brings P to indifference given she cannot implement her overall optimal policy. This allowed us to derive the acceptance sets A_D and A_A

and given those we derived optimal C's behaviour and confirmed the conjecture correct.

Using the theorem just given it follows that taking the conjectured A_D and A_A if we derive a solution to the system of Bellman equations above, the solution is unique. Hence the equilibrium constructed above must be unique.

The essential adjective comes from the fact that we have proven uniqueness in the class of equilibria where C brings P to indifference given she cannot implement her overall optimum. But there might be other equilibria where this feature does not hold. Yet another reason to add the essential adjective is the fact that we have proven uniqueness of the value functions, not uniqueness of the proposal strategies. However, it is easy to see that the offer strategies are unique solutions to the C's optimization problem.

A2 Proof of proposition 2

For the first part, notice that inflation possibly stays outside J only if every period is a D period. Probability of path of n periods all of them being D ones is p_d^n which goes to zero.

For the second part, notice $p_A(x) = x \Rightarrow p_D(x) = x$. Hence we need to find a set of x for which $p_A(x) = x$ holds. This is given by $\langle \pi^* - \phi\delta p_d, \pi^* + \phi\delta p_d \rangle$ which has the width indicated.

For the third part, it is apparent that the set of x for which $p_A(x) = \pi^*$ is a finite collection of $\{x_1, x_2, \ldots\}$ which has measure zero in X.

A3 Proof of proposition 3

Assume there exists an equilibrium with $p_A(x) = \pi^* + \varepsilon$ for some $x \in X$ and $\varepsilon > 0$. Denote by γ the equilibrium policy $\{p_A(x) = \pi^* + \varepsilon, q_D(x)\}$ and by γ' policy $\{\pi^* + \varepsilon/2, q_D(x)\}$. By the definition of the equilibrium it must be that γ solves C's problem, that is, it is a solution to

$$\max_{\{p,q\} \in A_A(x)} \left\{ -(p - \pi^*)^2 + \delta V_C(q) \right\}$$
$$\text{s.t. } -(p - \pi^*)^2 + \delta V_P(q) \geq -(x - \pi^*)^2 + \delta V_P(x).$$

By continuity of the constraint in p the policy $\gamma' \in A_A(x)$. C's utility from γ' is $-\varepsilon^2/4 + \delta V_C(q_D(x))$ and from γ it is $-\varepsilon^2 + \delta V_C(q_D(x))$. By assumption γ is an equilibrium hence

$$-\varepsilon^2 + \delta V_C(q_D(x)) \geq -\varepsilon^2/4 + \delta V_C(q_D(x))$$

which implies $\varepsilon^2 \leq \varepsilon^2/4$, a contradiction. As we cannot be sure about the existence of the equilibrium yet, the existence qualification must be added to the proposition 3.

A4 Proof of proposition 4

We establish the result using the series of claims.

claim 15. *Let $X^- = X \setminus (\pi^* - \phi, \pi^* + 3\phi)$ and $z, z' \in X^-$. For any $x \in X^-$ the equilibrium is given by*

$$\begin{aligned} q_A(x) &= z \qquad & p_A(x) &= \pi^* \\ q_D(x) &= z' \qquad & p_D(x) &= \pi^* - \phi. \end{aligned}$$

where the inflation strategies are unique. Moreover, for any $x \in X^-$, $V_C(x) = 0$ and $V_P(x) = -\frac{4\phi^2 p_d}{1-\delta}$.

Proof. We first show $\rho = \{q_D(x) = q_A(x) = x, p_D(x) = \pi^* - \phi, p_A(x) = \pi^*\}$ is an equilibrium for any $x \in X^-$. Fix $x \in X^-$. Note that $\{p_D(x) = \pi^* - \phi, x\} \in A_D(x)$ and $\{p_A(x) = \pi^*, x\} \in A_A(x)$ and both increase C's utility compared to $\{x, x\}$. It also follows ρ induces $V_C(x) = 0$ hence C clearly cannot do better. Therefore ρ is an equilibrium.

Having the equilibrium for given x, notice it induces the same path of inflation decisions for a fixed path of A and D periods as any $x' \in X^-$. It follows $V_C(x)$ and $V_P(x)$ must be constant on X^-. Therefore the first part of the claim follows.

To show uniqueness of the inflation offers notice C's utility strictly decreases by offering anything other than inflation specified in the claim.

The fact that $V_C(x) = 0 \ \forall x \in X^-$ follows immediately from two previous remarks. To show $V_P(x) = -\frac{4\phi^2 p_d}{1-\delta}$ using the constancy of $V_P(x)$ we can write

$$V_P(x) = p_d[-4\phi^2 + \delta V_P(x)] + (1 - p_d)[\delta V_P(x)]$$

which after rearrangement gives $V_P(x)$ in the claim. □

claim 16. *Let $X^+ = (\pi^* - \phi, \pi^* + 3\phi)$. Then for all $x \in X^+$, $V_C(x) < 0$.*

Proof. Assume there exists an equilibrium such that $V_C(x) = 0$ for some $x \in X^+$. It follows $V_P(x) = -\frac{4\phi^2 p_d}{1-\delta}$. Take D period, if P rejects today and follows the equilibrium strategy from then on his utility is $-(x - \pi^* - \phi)^2 - \frac{4\phi^2 \delta p_d}{1-\delta}$ whereas if he accepts (as equilibrium demands) his utility is $-4\phi^2 - \frac{4\phi^2 \delta p_d}{1-\delta}$. For this to be an equilibrium it must be that

$$-(x - \pi^* - \phi)^2 - \frac{4\phi^2 \delta p_d}{1 - \delta} \leq -4\phi^2 - \frac{4\phi^2 \delta p_d}{1 - \delta}$$

which rewrites as $(x - \pi - \phi)^2 \geq 4\phi^2$ and holds for $x \notin (\pi^* - \phi, \pi^* + 3\phi)$, a contradiction to $x \in X^+$. □

claim 17. *C's offer γ in both types of periods, provided she cannot implement her overall optimum, makes P indifferent between γ and the default policy $\bar{\gamma}$.*

Proof. Denote by $\{p_i^*, q_i^*\}$ for $i \in \{A, D\}$ C's most preferred policy. It follows $p_A^* = \pi^*$, $p_D^* = \pi^* - \phi$ and $q_i^* = \arg\max_{x \in X} V_C(x)$. Fix $x \in X$ and assume $\{p_i^*, q_i^*\} \notin A_i(x)$ (notice this implies $x \in S \subseteq X^+$).

$\mathbf{i = D}$ Take some $\{p_D(x), q_D(x)\}$ and assume it is part of an equilibrium and that it is in the interior of $A_D(x)$. By the continuity of the P's acceptance condition in p, it follows $\{p_D(x) - \varepsilon, q_D(x)\}$ for some $\varepsilon > 0$ would be accepted as well and would make C better off. It follows $\{p_D(x), q_D(x)\}$ cannot be an equilibrium.

$\mathbf{i = A}$ We already know $p_A(x) = \pi^*$ for all $x \in X$. Take some $\{\pi^*, q_A(x)\}$ and assume it is part of an equilibrium and that it is in the interior of $A_A(x)$. Take some q that belongs to the boundary of $A_A(x)$. As $\{p_A^*, q_A^*\}$ is not in $A_A(x)$ it is clear such q must exist.

Now as $q_A(x)$ is in the interior of $A_A(x)$ and q on its boundary, it follows $V_P(q_A(x)) > V_P(q)$ and

$$\begin{aligned} A_D(q_A(x)) \subseteq A_D(q) \Rightarrow &\quad U_D(q_A(x)) \leq U_D(q) \\ A_A(q_A(x)) \subseteq A_A(q) \Rightarrow &\quad U_A(q_A(x)) \leq U_A(q). \end{aligned}$$

Summing up the two inequalities multiplied by p_d and $1 - p_d$ respectively gives

$$p_d[U_D(q_A(x)) - U_D(q)] + (1 - p_d)[U_A(q_A(x)) - U_A(q)] \leq 0.$$

At the same time $V_C(q_A(x)) > V_C(q)$ since $q_A(x)$ is chosen (equality in general is possible, but then we might simply assume C chooses q instead). Rewriting $V_C(q_A(x)) > V_C(q)$ gives

$$p_d[U_D(q_A(x)) - U_D(q)] + (1 - p_d)[U_A(q_A(x)) - U_A(q)] > 0$$

a contradiction. □

claim 18. *If P is not brought to indifference in A period for some x, then*

$$p_A(x) = \pi^* \qquad q_A(x) = z$$

for some $z \in X^-$.

Proof. Note that converse of claim 17 reads if P is not brought to indifference then C can implement her overall optimum, which by the claims 15 and 16 is the pair indicated. □

claim 19. *For any $x \in X^+$ if P is brought to indifference in A period for default policy x, then he is brought to indifference in D period for the same default policy.*

Proof. We prove the converse, i.e. if P is not brought to indifference in D period, then he is not brought to indifference in A period.

Note that by claim 17 if P is not made indifferent, then C can implement her overall optimum. For the D period this is $\{\pi^* - \phi, z\}$ with $z \in X^-$. This implies

$$-(x - \pi^* - \phi)^2 + \delta V_P(x) \leq -4\phi^2 + \delta V_P(z)$$

which after rearrangement gives

$$-(x-\pi^*)^2 + \delta V_P(x) \leq \delta V_P(z) - [3\phi^2 + 2\phi(x-\pi^*)]$$

where the term in the square brackets is positive for any $x \in X^+$. It then follows $\{\pi^*, z\} \in A_A(x)$. $\square$

Using the claims above, we can compute the continuation value function for P. For $x \in X^-$ it is given by

$$V_P(x) = p_d[-4\phi^2 + \delta V_P(x)] + (1-p_d)[\delta V_P(x)].$$

For $x \in X^+$ for which P is brought to indifference in A and D periods it is

$$V_P(x) = p_d[-(x-\pi^*-\phi)^2 + \delta V_P(x)] + (1-p_d)[-(x-\pi^*)^2 + \delta V_P(x)].$$

Finally, for $x \in X^+$ for which P is brought to indifference only in D periods it is

$$V_P(x) = p_d[-(x-\pi^*-\phi)^2 + \delta V_P(x)] + (1-p_d)[\delta V_P(z)]$$

where $z \in X^-$. After some rearrangement, the equations above are those given in the proposition. It is then straightforward to establish the intervals over which those apply.

A5 Proof of proposition 5

We first establish properties of the acceptance correspondences A_D and A_A to be used later.

claim 20. *For any $x \in X$ the acceptance correspondences $A_D(x)$ and $A_A(x)$ are nonempty, compact valued and upper-hemicontinuous.*

Proof. The nonempty and compact valued parts of the claim follow by definition. To prove upper-hemicontinuity of the acceptance correspondence

$$A_D(x) = \{(p,q) \in X^2 | -(p-\pi^*-\phi)^2 + \delta V_P(q) \geq -(x-\pi^*-\phi)^2 + \delta V_P(x)\}$$

denote $\mathbf{x} = (x,x)$, $\mathbf{p} = (p,q)$ and $f(\mathbf{p}) = -(p-\pi^*-\phi)^2 + \delta V_P(q)$.

Pick two sequences $\{\mathbf{x}_\alpha\} \to \mathbf{x}$ and $\{\mathbf{p}_\alpha\} \to \mathbf{p}$ such that $\mathbf{p}_\alpha \in A_D(x_\alpha)\ \forall\alpha$. Note that by non-emptiness of A_D this can be done. We need to show $\mathbf{p} \in A_D(x)$.

Suppose $\mathbf{p} \notin A_D(x)$. Then

$$f(\mathbf{x}_\alpha) \leq f(\mathbf{p}_\alpha)\ \forall\alpha$$
$$f(\mathbf{x}) > f(\mathbf{p}).$$

Summing the two inequalities gives

$$f(\mathbf{x}_\alpha) - f(\mathbf{x}) < f(\mathbf{p}_\alpha) - f(\mathbf{p})\ \forall\alpha.$$

Taking the limit for $\alpha \to \infty$ on both sides gives contradiction to continuity of $f(\cdot)$. For A_A the proof is analogous and hence omitted. $\square$

To prove the main proposition, we need to show existence and uniqueness of the solution of the system of functional Bellman equations

$$U_D(x) = \max_{\{p,q\}\in A_D(x)} \{-(p - \pi^* + \phi)^2 + \delta p_d U_D(q) + \delta(1 - p_d)U_A(q)\}$$
$$U_A(x) = \max_{\{p,q\}\in A_A(x)} \{-(p - \pi^*)^2 + \delta p_d U_D(q) + \delta(1 - p_d)U_A(q)\}.$$

We already know the acceptance correspondences of the system are upper-hemicontinuous. Hence we can use the same theorem 1 as in proof of proposition 1 above where the remaining conditions are clearly satisfied. Similar argument also proves the existence of coincidence solution U_D^*, U_A^* which solves the system of Bellman functional equations as a whole.

Finally, the essential uniqueness comes from the fact that the uniqueness applies only to the value function, not to the equilibrium proposal strategies. Indeed claim 15 implies that for the default policies $x \in X^-$ there is continuum of equilibria. However, each of them gives rise to the identical C's value function.

A6 Proof of proposition 6

For the first part we are looking for a set of x such that $q_D(x) = q_A(x) = x$ holds. Focusing on the A periods and $x \in X^+$ from the proposition 3 and claim 17 above we are looking for the solution to the equation

$$-(x - \pi^*)^2 + \delta V_P(x) = \delta V_P(q_A(x))$$

where $q_A(x) = x$. It is immediate that the only solution to this equation is $x = \pi^*$ which has measure zero.

For the $x \in X^-$ we know by claim 15 above that $q_D(x) = z$ and $q_A(x) = z'$ with $z, z' \in X^-$. It is clear that we can set $z = z' = x$. As we are allowed to do so only for $x \in X^-$, the measure of the set for which $q_D(x) = q_A(x) = x$ is at most measure of X^-, which proves the first part.

To show the second part, we know that the largest J we can obtain is $X^- \cup \{\pi^*\}$. For X^- we know by claim 15 that $p_D(x) = \pi^* - \phi$ and $p_A(x) = \pi^*$. Hence the only remaining possibility is that $q_D(\pi^*) = \pi^*$ which proves the second part.

Finally the third part follows directly from proposition 3.

A7 Proof of proposition 7

We prove two claims that together prove the proposition. The strategy of the proof borrows heavily from Riboni and Ruge-Murcia (2008).

claim 21. *The difference in utilities associated with two sequences of policy decisions is linear in ϕ.*

Proof. Take two general sequences of inflation decisions $\mathbf{p} = \{p_0, p_1, \ldots\}$ and $\mathbf{p}' = \{p_0', p_1', \ldots\}$. Utility associated with those inflation sequences for committee member with preference parameter ϕ is

$$U(\mathbf{p}, \phi) = -\sum_{t=0}^{\infty} \delta^t (p_t - \pi^* - \phi I(D_t))^2$$

where $I(D_t)$ is D period indicator function. Taking the derivative of the difference $U(\mathbf{p}, \phi) - U(\mathbf{p}', \phi)$ with respect to ϕ gives

$$\frac{\partial [U(\mathbf{p}, \phi) - U(\mathbf{p}', \phi)]}{\partial \phi} = \sum_{t=0}^{\infty} 2\delta^t I(D_t)(p_t - p_t')$$

which does not depend on ϕ. It follows the difference in utility between $\mathbf{p}$ and $\mathbf{p}'$ is linear in ϕ. □

Next claim shows that the proposal is passed if and only if it is accepted by the median member. Formally, for the committee of N (N odd) members denote their preference parameters $\{\phi_1, \ldots, \phi_N\}$ such that $\phi_i < \phi_j$ for every pair $1 < i < j < N$. Then the median member has the preference shock ϕ_m which satisfies $|\{\phi_i | \phi_i > \phi_m\}| = |\{\phi_i | \phi_i < \phi_m\}|$.

claim 22. *Assuming stage-undominated voting strategies, for a committee with N members with N odd, C's proposal γ is passed if and only if it is accepted by the median committee member.*

Proof. For sufficiency, assume median member accepts, then by the preceding claim either all committee members with $\phi_i > \phi_m$ accept or all committee members with $\phi_i < \phi_m$ accept. In either case, γ passes.

For necessity, assume median member does not vote for γ. Then either all members with $\phi_i > \phi_m$ do not vote for γ or all members with $\phi_i < \phi_m$ do not vote for γ. In either case γ is not approved. □

Using the claim 22 and the fact that the median preserving committee expansion leaves the identity of the median voter unchanged, it follows C's proposal strategies have to be identical to the model with only two committee members. And since C takes into account only presence of the median voter when deciding about her proposal strategy, all her proposals are passed in equilibrium. Hence the proposition follows.

A8 Numerical estimation of the equilibrium

This section describes the procedure to obtain numerical estimates of the equilibrium in the model with the directive. We use standard value function approximation method.

First of all recall that by proposition 3 we know $p_A(x) = \pi^*$. Furthermore, from proposition 4 we know the shape of the acceptance sets and equilibrium offers for $x \in X^-$ and for some $x \in X^+$ in A periods when C is able to implement her overall optimum. Finally from proposition 5 we know the equilibrium is unique.

To estimate the remaining part of the equilibrium, we restrict the policy space to $X = \langle \pi^* - 1.1\phi, \pi^* + 3.1\phi \rangle$ and specify grid of discrete nodes $\{d_1, \ldots, d_N\} \in X$. Call this grid G. In practice we used $\pi^* = 2, \phi = 1$ which with the distance of the neighbouring nodes equal to 0.001 gave $N = 4201$. We also experimented with different values of π^* and ϕ but the shape of the equilibrium is not affected as π^* only 'shifts the equilibrium up and down' the vertical axis and ϕ only 'stretches the equilibrium' between $\pi^* - \phi$ and $\pi^* + 3\phi$.

With the policy space specified, we follow the following iterative procedure. At the iteration t we solve C's optimization problem for A and D periods for each default policy in G. Denote by $V_C^t(G)$ the $N \times 1$ vector of C's continuation values, each of them associated with a distinct node $d_i \in G$ at the t-th step of the iteration.

For D periods we solve for each $d_i \in G$

$$\max_{\{p,q\} \in A_D(d_i)} -(p - \pi^* + \phi)^2 + \delta V_C^t(q)$$

by searching the grid G. This gives us two $N \times 1$ vectors of equilibrium offers for the D period, call those $\mathbf{p}_D^t$ and $\mathbf{q}_D^t$.

For A periods we already know $p_A(x) = \pi^*$ hence for each $d_i \in G$ we solve

$$\max_{\{\pi^*,q\} \in A_A(d_i)} V_C^t(q)$$

again by searching the grid G. This gives us one $N \times 1$ vector of status-quo offers for the A period, call it $\mathbf{q}_A^t$.

Finally we compute the $N \times 1$ vector of C's continuation values

$$V_C^{t+1}(G) = p_d \left[-(\mathbf{p}_D^t - \pi^* + \phi)^2 + \delta V_C^t(\mathbf{q}_D^t) \right] + (1 - p_d) \left[\delta V_C^t(\mathbf{q}_A^t) \right]$$

and proceed to the iteration $t+1$. As usual, for the first step of the iteration we used $V_C^1(G) = \mathbf{0}$. In practice the rate of convergence of the results is very fast and the offer strategies become almost indistinguishable between the iterations from around $t = 10$ on. Nevertheless the estimation presented is based on $t = 30$ and we also experimented with iterations up to $t = 10.000$ to be sure about the results.

The reason why we use this rather rudimentary numerical procedure instead of some more involved one (e.g. better optimization algorithm and functional approximation for V_C) is twofold. First, we suspected the V_C to be ill-behaved with number of local maxima and we did not want the optimization algorithm to pick a wrong one especially as the acceptance sets are in general not convex. Second, we suspected the resulting equilibrium to involve several discontinuities and we did not want the functional approximation to 'smooth out' the problem.

We also experimented with the different estimation procedures. The first one involved estimating the full model without specifying the acceptance sets. The difference is that instead of having same A_A and A_D in each step, we started with $V_P^1(G) = 0$ and derived new V_P at each step of the iteration in a similar way as the V_C. This gave us new acceptances sets for the next iteration. As this procedure gives almost identical results whereas the one presented above takes only a fraction of time, we use the faster one.

We also tried to estimate the equilibrium using the functional approximation of the V_C function. We used cubic splines doubling the nodes at the values where we expected kinks in the V_C function but the results were again nearly identical.