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PURCHASING PUBLIC GOODS AT AUCTION UNDER MAJORITY RULE

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Abstract

This study suggests a two-stage mechanism whereby a small group of actors (e.g. firms) collects contributions to purchase a discrete, excludable good which they intend to use collectively in non-rival ways. This good is assumed to be offered at an auction, where other actors might be bidding to get it for sole use. Economic theory has offered sophisticated mechanisms to implement an efficient allocation of public goods, but the proposed solutions are frequently rather complicated and impractical for producing plausible mechanisms. This study presents a mechanism which tries to meet the requirements of simplicity. Actors who agree to place joint bids - for a good that they will share *ex post* - must contribute at least a minimum fraction of their wealth. We show that, under majority rule, this fraction can be set using a translation to our circumstances of the median-index theorem, which applies under weak assumptions. Thus, in the first stage of our mechanism, participants bidding for shared use vote on a minimum percentage of wealth to pay (which is common knowledge); in the second stage, they bid at least that minimum amount, then the auctioneer compares the total of these bids with the highest bid (if any) for exclusive use, and provisionally assigns the lot to individual or shared use accordingly. This continues till there is no excess demand.

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1.- Introduction: bidding for (public) goods

This study considers a setting where a discrete good - which can be used either collectively by several actors in non-rival ways or individually - is available at an auction. Bidding for a (non-rival) good as a group of players seems to present at least one crucial issue, compared to circumstances where everyone is bidding against the others, because players have to use a mechanism to set their individual contributions to a jointly submitted bid. The aim of this work is to suggest and investigate a plausible way to form an aggregate bid for a good at auction. The good will be used collectively or individually according to the offers received by the auctioneer. For instance, the good might be an input that could be shared by its owners if purchased collectively, or, alternatively, used by a sole owner. In both cases, the group or individual who submits the auction winning bid will enjoy exclusive use of the good. In this analysis, it is assumed that grouped players agree on usage terms before the auction takes place (e.g., by setting usage standards); therefore, this work does not deal with the problem of optimal usage of a public good, which usually involves an exogenously given amount of it.

The mechanism suggested requires auction participants who bid collectively to contribute at least a minimum fraction of their wealth; this fraction is set by majority vote (each player has one vote). We show that, under majority rule, the minimum fraction of wealth can be set using a translation to our circumstances of the median-index theorem, which applies under weak assumptions (Cave and Salant 1995). Thus, our envisaged mechanism is such that, in the first stage, players who will submit a joint bid vote on a common minimum percentage of their wealth to pay; in the second stage, individual participants submit their individual offers, while group participants contribute at least the minimum amount voted in the previous stage and collect an aggregate bid. Then, the auctioneer compares this bid with the highest bid (if any) for sole use, and provisionally assigns the lot auctioned off accordingly. This continues till there is no excess demand.

The rest of this paper is organized as follows. Section 2 provides a review of the literature on the public good provision problem, with focus on voluntary contribution games. Section 3 introduces our basic model, which is a translation of Cave and Salant's model on cartel quotas under majority rule to our setting (Cave and Salant 1995). Section 4 explores that translation by verifying, firstly, that an analogous set of properties is satisfied and, secondly, that the median-index theorem applies – *mutatis mutandis* - to our setting. Section 5 closes this paper.

2.- Literature review

Two important features of the public goods provision problem are (i) the mechanism or institution for providing the good and (ii) the incentive structure for potential providers (Isaac *et Al*. 1989, p. 217). Economic theory has suggested sophisticated mechanisms to implement an efficient allocation of public goods (for surveys, see: Green and Laffont 1977; Groves and Ledyard 1987; Laffont 1987; Cullis and Jones 1998). The general results are technically impressive, but the proposed solutions are frequently rather complicated and, hence, impractical for producing plausible mechanisms which induce efficient contributions to public goods (see criticisms in Walker 1981, p. 71; Laffont 1987, p. 567; Jackson and Moulin 1992, p. 2; Falkinger *et Al*. 2000, p. 247). Therefore, several authors have suggested incentive mechanisms which seem to meet the requirements of simplicity (Falkinger *et Al*. 2000).[1] Among those mechanisms, our study is more closely related to analyses of settings where the public good is provided by voluntary rather than by compulsory action. For instance, Jackson and Moulin (1992) study circumstances - with characteristics similar to our problem - where several agents must decide whether or not to undertake a project that will benefit them all and be consumed without rivalry by all agents. Their mechanism uses no statistical information about the distribution of agents' characteristics and places the burden of acquiring information about the preference profile upon the agents themselves. In their model, Jackson and Moulin assume that individuals know their own valuations and that at least two agents know the average of all the agents valuations. The authors note that "[t]his is admittedly a strong assumption, but it appears necessary if we wish to use simple, intuitive mechanisms [...]. The alternative route, relying on the existence of Bayesian beliefs about mutual preferences, has a stronger claim to realism as a model of individual behaviour, but its mechanisms are pegged to the Bayesian characteristics of a particular group of agents" (p. 126). In addition, it is noted that mechanisms which require statistical information about agent's valuations are not very meaningful among a few agents (as in the circumstances that we consider).

A different theoretical approach in the analysis of voluntary games is proposed by Bagnoli and Lipman (1989), who develop a contribution game for the provision of non-excludable public goods. They show that a natural game – in contrast to quite complex games of full implementation – in fact fully implements the core of their economy in undominated perfect equilibria. In their model, individual agents have sufficient incentives to voluntary achieve

[1] For some of those mechanisms, e.g. the compensation mechanism (Varian 1994) and the Falkinger mechanism (Falkinger 1996), their effectiveness has been tested in experiments (Bracht *et Al*. 2008).

the Pareto efficient outcome. Moreover, Bagnoli and Lipman's theory of a voluntary contribution game was tested in a laboratory setting by Bagnoli and McKee (1991), who found that differences in wealth or valuation do not bear on the capability of providing the public good (even if those differences are considerable). Another relevant feature of Bagnoli and Lipman`s work is that they consider situations where a public good could be produced only if the sum of contributions met or exceeded some threshold, which is known by participants. In McBride (2006), the case of discrete public goods provision under threshold uncertainty is analyzed. The author shows that, for a large class of threshold probability distributions, equilibrium contributions will be higher under increased uncertainty if the public good's value is sufficiently high.

Other studies of circumstances similar to ours refer to alternative environments, sometimes said to exhibit the assurance problem, where a potential provider can have an incentive to contribute "if, and only if, he or she has a credible guarantee that others will also contribute. Absent such a guarantee, the provider may withhold" (Isaac *et Al*. 1989, p. 217). For instance, Boadway *et Al*. (2007) focus on a multi-stage process of non-cooperative voluntary provision of public goods, where, in the first stage, one or more players announce contributions that may be conditional on the subsequent contributions of others; in later stages, players choose their own contributions and fulfill any commitments made in the first stage. They find that efficient levels of public goods can be achieved under some circumstances, while in others commitment is ineffective.

Some features of those environments may be found in voluntary collective action approaches to the optimal provision of public goods, based on mutual subsidization by agents of their individual contributions. The aim is to cope with the free-rider problem and to increase voluntary contributions to the provision of a public good above Nash-Cournot levels by matching behaviour, which was originally suggested by Joel Guttman in 1978 (Guttman 1978, 1987). Matching behaviour is "a strategy that makes an agent's contribution to the provision of a public good conditional on the contributions of his counterparts in order to induce them to contribute as well" (Guttman 1986, p. 172). Guttman's setting is a two-stage non-cooperative game: the first stage of the game is played to choose simultaneously the matching rates (i.e. the rate at which each player *i* will subsidize the sum of the flat contributions offered by the other players in the next stage); given the matching rates chosen in the first stage, the second stage of the game is played to determine, again simultaneously, the autonomous flat contributions. The model predicts Pareto optimal provision of a non-excludable public good by identical actors with perfect information, regardless of the number of actors and by two non-identical actors. Guttman's model offers interesting theoretical results. However, implementation is difficult, because the model is based on a two-stage game that is hard to play effectively, especially when the number of players grows above a few ones.

The issue of participation is underlined in Palfrey and Rosenthal (1984), who argue that, in the analysis of the provision of public goods, the free-rider problem leads to two theoretical issues: (i) the demand revelation question and (ii) participation. In their model, to isolate the participation problem from the demand revelation problem, they deliberately make all players identical and thereby remove all incentives for players to conceal preferences. The only question is whether the group can achieve an optimum by voluntary contributions. Their analysis investigates the minimum contributing set (MCS), where individuals may choose to contribute all of their wealth, or not, to the provision of a public good, under the rule that the good will be supplied if a pre-announced number of individuals contribute. In the MCS, however, the announced number is smaller than the entire group; hence, some individuals will not contribute, but still be able to use the public good (if provided). Van de Kragt *et Al*. (1983) and Dawes *et Al*. (1986) found that, in their laboratory settings, the MCS regime is largely successful in generating an efficient outcome.

3.- Basic model

Our basic model has a fundamental element in a tentative translation of an earlier model developed by Cave and Salant (1995) on cartel quotas under majority rule; we propose a translation of that model to the circumstances that we are investigating – i.e. collecting an aggregate bid to purchase a (public) good - and we use the following notation:

N number of players (i.e. firms) bidding together as a group;

w_i bidder *i*'s (non-negative) exogenous wealth, which is assumed to be common knowledge and immediately convertible in assets accepted by the seller – i.e. the auctioneer - at no cost (w_i is a firm-specific scalar used to translate F into a restriction on *i*);

c_i constant cost of capital (opportunity cost of funds) for player *i*;

b_i 'individual' bid, i.e. the amount of assets offered as individual contribution to the purchase of a (public) good;

B_{-i} sum of individual contributions offered by the n players, excluding player *i*, i.e. $B_{-i} = \sum_{j \neq i} b_j$;

F minimum fraction (or percentage) of wealth to pay, chosen by the N bidders by unweighted majority-rule voting; hence F will be the same for every player. F is a committee's prior choice and if the committee chooses the fraction F, then firm *i*'s contribution must be no lower than Fw_i.

In our model, firms are indexed in order of ascending cost of capital; if two firms have the same cost of capital, firms are indexed in order of increasing wealth:

if $c_i > c_j$ or

if $c_i = c_j$ and $w_i > w_j$

then $i > j$

(if $c_i = c_j$ and $w_i = w_j$ then assign indexes arbitrarily).

Firms bidding collectively are assumed to spend their wealth on contributions to the purchase of a public good input. To avoid free riding, those firms must join a committee (i.e. the cartel in the original model), whose fundamental task is to vote by majority on the minimum fraction F of individual wealth that must be contributed.

Let $B = \sum_1^N b_i = B_{-i} + b_i$ denote the aggregate bid; then $B \geq \sum_1^N Fw_i = F \sum_1^N w_i$.

Also, let f(B) denote average benefits (e.g. revenues) attainable by winning aggregate bids B.

Firm *i* is assumed to maximize profits. For this purpose, *i* has to choose its preferred contribution (b_i) to the collective project, given the contributions offered by other firms and the previously selected fraction F (where $F \in [0, 1]$):

$$\max \quad b_i\,[f(B) - c_i] = b_i\,[f(B_{-i} + b_i) - c_i]$$

s.t.

$$b_i \geq F\,w_i.$$

Each bidder's equilibrium profit π_i depends, *inter alia*, on the prior choice of F; hence we also regard them as induced profits π_i(F). In addition, in our setting equilibrium profits will be zero for every group bidder if B is less than max{ L_k }, where L_k is the amount of money offered by firm *k* who is bidding against everyone else to get the good auctioned off for sole use (*k* is not in the group of n firms bidding for shared use of the good and therefore is not in the voting committee):

$$\pi_i = 0 \qquad \text{if} \qquad B < L_k \qquad \forall i \in [1, N], k \notin [1, N].$$

Assume total benefits TB depend on the amount of funds collected in the following way:

$$TB = b_i \cdot [f(B_{-i} + b_i)] = b_i \cdot f(B).$$

Then marginal benefits MB are equal to f(B) + b_i f′ (B). Assume MB is strictly decreasing. With MB strictly decreasing, f(B) decreases. However, π_i (F) will be positive as long as f(B) exceeds c_i.

4.- Analysis of the model

4.1.- Economic equilibrium which would result if the group had voted for any arbitrary fraction F of wealth

Assume that:

- the average revenue function f(B) is strictly decreasing and twice continuously differentiable;
- the total benefit function [i.e., b_i * f(B)] is strictly concave;
- if $b_i \to \infty$ then lim f(B) = 0;

- there would be positive profits if the lowest-cost firm contributed funds whose cost of capital is c_1 [i.e., $f(B) - c_1 > 0$].

Given those benefit assumptions, then a unique Cournot equilibrium - induced by any given fraction F of wealth - exists in pure strategies.

Proofs.

Existence and uniqueness are proved in Appendix A.

The equilibrium is characterized by an aggregate bid (B) divided into a vector of individual bids ($b_1, b_2, .., b_N$) satisfying one of the following conditions for i = 1, 2, .., N:[2]

a) unconstrained participant: $f(B) + b_i f'(B) - c_i = 0$ and $0 < Fw_i < b_i$;

b) constrained participant: $b_i = Fw_i$ and $f(B) + Fw_i f'(B) - c_i \leq 0$

(firm *i* would like to contribute less than $b_i = Fw_i$, but – since *i* joined the procedure – it must contribute at least a minimum amount of funds, according to F).

4.2.- Fraction of wealth which a group would select under majority rule: the median-index theorem revisited

Assume voters (i.e. firms) are foresighted and self-interested. We want to prove that the median-index theorem (Cave and Salant 1995) applies to our setting.

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[2] If a firm would like to get access to a spectrum commons, but expects average benefit to be so low that it would not be able to make positive profits, it will not join the procedure and will remain an outsider; of course, $b_o = 0$.

The median-index theorem states that – assuming an odd number of voters (N) are to select an alternative from a compact one-dimensional set of alternatives by simple majority rule – every ideal point of the firm with the median index on the committee will be weakly preferred to any other point by a majority of the voters, if the following preference assumptions are met:

1.- continuity, i.e. each voter's preferences can be represented by a continuous real-valued function on the set of alternatives;

2.- unconstrained monotonicity, i.e. each voter's preference function is monotonically decreasing above its cutoff[3];

3.- nesting of cutoffs and partial agreement, i.e. if voters are indexed so that someone with a higher cutoff has a lower index, then the preferences of any two voters display partial agreement[4] below their cutoff points.

Therefore we have to prove – preliminary - that an analogous set of preference assumptions is satisfied in our setting. Given the benefit assumptions above, we will introduce a 'translation' of the regularity condition to our setting. Then, following Cave and Salant (1995), we will show that our benefit assumptions and regularity condition are sufficient for a set of preferences[5] to display the following properties:

a) nested cutoffs;
b) partial agreement;
c) unconstrained monotonicity;

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[3] In our setting, a "cutoff" is the wealth fraction which exactly induces firm *i* to contribute the amount of funds that firm *i* would freely choose to maximize its profits (i.e. the constraint is just binding). A unique cutoff is associated with each firm (the proof is in Appendix B).

[4] The agreement in preference is said to be partial when no restrictions are placed on the preference if the firm with the smaller index prefers the larger fraction (or vice versa). In contrast, the agreement is said to be complete when firms have the same marginal cost and must therefore rank the two fractions identically (Cave and Salant 1987).

[5] Cave and Salant (1995) start showing properties for the induced preferences and then examine the majority-rule voting behaviour of any set of agents whose preferences satisfy a generalization of those properties. Profit functions describe our (induced) preferences.

d) continuity.

This will allow us to prove the existence of a Condorcet winner, that is a fraction of wealth to pay which will be selected by some majority of the firms. We will then consider uniqueness of a Condorcet winner.

Let B(F) denote the aggregate equilibrium bid induced by a majority decision to contribute fraction F of wealth. Also, let F_j denote the fraction that would just bind on firm *j*, i.e. *j*'s marginal cost and benefit are equal for $F = F_j$. Thus, given $B_{-i}(F_j)$, F_j is implicitly defined as

$f(B(F_j)) + F_j w_j f'(B(F_j)) - c_j = 0$.

F_j is regarded as *j*'s "cutoff", because it is the fraction which exactly corresponds to the amount of *j*'s wealth that *j* would bid to maximize its profits – whereas, above that fraction, *j* has to contribute more than the amount where its marginal benefit equals marginal cost (*j*'s profit maximization is constrained).

<u>Regularity condition</u>

In Cave and Salant's model, the regularity condition is a crucial one: "[it] is necessary and sufficient for the cutoffs to be nested and […] is sufficient for the existence of a Condorcet quota. When cutoffs are nested, the induced preferences display a property we refer to as 'partial agreement'" (Cave and Salant 1995, 87); in addition to nesting and partial agreement, the preferences display "continuity" and "unconstrained monotonicity" (Cave and Salant 1995, 88).

We will therefore elaborate an analogous regularity condition, for the circumstances that we are investigating. Assume that the following regularity condition holds for each pair of firms *i* and *j* such that i > j:

$f'(B(F_j)) F_j (w_i - w_j) \leq c_i - c_j$.

This is a reduced form of

$f(B(F_j)) + f'(B(F_j))\, F_j\, w_i - f(B(F_j)) - f'(B(F_j))\, F_j\, w_j \leq c_i - c_j$

where – given the selected fraction $F = F_j$ - the first part of the left-hand side is the marginal benefit for firm *i* when firm *i* bids the minimum amount required by the committee ($b_i = F_j\, w_i$); while the second part of the left-hand side is the marginal benefit for firm *j* (*j* would bid exactly that fraction of its wealth which is required by the committee). The right-hand side is the difference in the costs of capital for firms *i* and *j*.

Then any fraction binding on one firm must also bind on firms with greater indexes: for instance, if F_j is a fraction binding on firm *j* and $i > j$, then F_j must also be binding on firm *i*. In fact, if F_j is binding on *j*, when $F = F_j$ marginal benefit and marginal cost are equal for firm *j*, but firm *i* would be better off with a fraction F lower than F_j (i.e. $F < F_j$), because when $F = F_j$ firm *i*'s marginal benefit are lower than its cost of capital. Nevertheless it must contribute at least $F_j\, w_i$. This can be shown by re-writing the regularity condition in the following way:

$f(B(F_j)) + f'(B(F_j))\, F_j\, w_i - c_i \leq f(B(F_j)) + f'(B(F_j))\, F_j\, w_j - c_j$

and, given that F_j is just binding on *j*, $f(B(F_j)) + f'(B(F_j))\, F_j\, w_j - c_j = 0$; therefore

$f(B(F_j)) + f'(B(F_j))\, F_j\, w_i - c_i \leq 0$

which shows that, when $F = F_j$, for firm *i* marginal benefits are lower than its cost of capital (or, if equality holds, F_j is just binding on *i* as well as on *j*).

If firms face the same cost of capital, but firm *i* has greater wealth than firm *j*, $c_i = c_j$ and $w_i > w_j$; the regularity condition therefore becomes:

$f'(B(F_j))\, F_j\, (w_i - w_j) \leq 0$.

This can be manipulated[6] to get

$f(B(F_j)) + f'(B(F_j))\, F_j w_i \le f(B(F_j)) + f'(B(F_j))\, F_j w_j = c_j = c_i$

which, again, shows that for $F = F_j$ and $c_i = c_j$ marginal benefits are lower than firm *i*'s cost of capital (or, if equality holds, F_j is just binding on *i* as well as on *j*).

It can be noted that, if firm *i* and firm *j* have the same wealth, $w_i = w_j$ (and $c_i \geq c_j$); then in the regularity condition

$f'(B(F_j))\, F_j\, (w_i - w_j) \le c_i - c_j$

the left-hand side is equal to zero; therefore the regularity condition holds (by assumption, $c_i - c_j \geq 0$).

The regularity condition also holds in applications where fractions are set equal to the Cournot-equilibrium individual contributions prior to the formation of a (voting) committee: this is the case where no minimum bid is required from each player, who can bid as little as he likes (the Cournot contribution).

Is the regularity condition sufficient for the existence of a Condorcet fraction? Cave and Salant (1995, 89) show that "any set of preferences displaying nested cutoffs, unconstrained monotonicity, partial agreement, and continuity must have a Condorcet winner". Therefore, to go on with the 'translation' of Cave and Salant's model, the average benefit assumptions, together with the regularity condition, should be sufficient for the set of induced preferences arising from the Cournot equilibrium to display the following properties:

1) nested cutoffs;
2) partial agreement;
3) unconstrained monotonicity;

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[6] Recall: $F = F_j$ which is the fraction just binding on firm *j*; hence marginal benefit and cost of capital are the same for firm *j*. Moreover, in this case it is assumed that the difference in the costs of capital is zero – i.e. firms face the same cost. Thus, for $F = F_j$ marginal benefit for firm *j* is also equal to firm *i*'s cost of capital.

4) continuity.

Those properties are translated below to our circumstances.[7]

1) Nested cutoffs.

If $i > j$, then $F_i \leq F_j$ for any couple of firms; hence cutoffs are nested:

$F_N \leq F_{N-1} \leq \ldots \leq F_2 \leq F_1$

(that is, if firms face different marginal costs of capital, when $i > j$ firm *i* has greater marginal cost than firm *j* - hence firm *i* prefers a fraction F lower than F_j; if marginal cost is the same for both firms, and firm *i*'s wealth is greater than firm *j*'s wealth, then again firm *i* prefers a fraction F lower than F_j)

The 'translated' regularity condition is necessary and sufficient for the cutoffs to be nested:

$f'(B(F_j))\, F_j\, (w_i - w_j) \leq c_i - c_j$ iff $F_N \leq F_{N-1} \leq \ldots \leq F_2 \leq F_1$.

Note that, by adding the implicit definition of F_j and the regularity condition, we obtain that also firm *i* is constrained at fraction F_j :

$f(B(F_j)) + F_j\, w_j\, f'(B(F_j)) - c_j$ (implicit definition of *j*'s cutoff)

$+ f'(B(F_j))\, F_j\, (w_i - w_j) - c_i + c_j$ (regularity condition)

$= f(B(F_j)) + F_j\, w_i\, f'(B(F_j)) - c_i \leq 0$ (*i* is constrained by fraction F_j).[8]

2) Partial agreement.

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[7] Proofs similar to those elaborated by Cave and Salant will be presented (some of those proofs are relegated to the appendixes).

[8] The implicit definition of *j*'s cutoff is $f(B(F_j)) + F_j\, w_j\, f'(B(F_j)) - c_j = 0$; the regularity condition is $f'(B(F_j))\, F_j\, (w_i - w_j) \leq c_i - c_j$ and it is a non-positive number; $f(B(F_j)) + F_j\, w_i\, f'(B(F_j)) \leq c_i$ shows that *i* is constrained at F_j.

If two fractions of wealth bind on each of two firms and one firm strictly prefers a particular fraction (case 2.a below) – or is indifferent between the two fractions (case 2.b below) - it is possible, in some circumstances, to deduce that the other firm likewise – and respectively - strictly prefers the same fraction, or weakly prefers one of the two fractions. "The agreement in preference is said to be 'partial' rather than 'complete' since no restrictions are placed on the preference if the firm with the smaller index prefers the larger [fraction] or, alternatively, if the firm with the larger index prefers the smaller [fraction]. In contrast, firms with identical marginal costs must rank the two [fractions] identically even in these cases. Agreement is then said to be 'complete'" (Cave and Salant 1995, 87).

2.a) Strict preference:

for any two firms *i* and *j* such that i < j and any pair of fractions ϕ and F such that $\phi < F \leq F_j \leq F_i$:

if $\phi \succ_i F$ then $\phi \succ_j F$

or

if $F \succ_j \phi$ then $F \succ_i \phi$.

"That is, if the firm with the smaller index strictly prefers the smaller [fraction], then so must the firm with the larger index; reciprocally, if the firm with the larger index strictly prefers the larger [fraction], then so must the firm with the smaller index" (Cave and Salant 1995, 87).[9]

Proof.

[9] If firm i strictly prefers F to ϕ and both fractions bind on i, then $F\, w_i \{f(B(F)) - c\} > \phi\, w_i \cdot$ $\cdot\{f(B(\phi)) - c\}$. We obtain $F\, w_j \{f(B(F)) - c\} > \phi\, w_j \{f(B(\phi)) - c\}$ by multiplying by the positive number w_j/w_i,. Therefore, if both fractions also bind on firm *j*, firm *j* strictly prefers F too.

Since $\phi \succ_i F$

$\phi\, w_i \{f(B(\phi)) - c_i\} > F\, w_i \{f(B(F)) - c_i\}$;

also, since $c_i \leq c_j$ and $\phi < F$

$-\phi\,(c_j - c_i) \geq -F\,(c_j - c_i)$.

Dividing the first inequality by w_i, adding the second weak inequality and multiplying by w_j, we obtain

$\phi\, w_j \{f(B(\phi)) - c_j\} > F\, w_j \{f(B(F)) - c_j\}$

which confirms that $\phi \succ_i F$.

The reciprocal statement can be verified *mutatis mutandis*.

2.b) Indifference:

for any two firms *i* and *j* such that $i < j$ and any pair of fractions ϕ and F such that $\phi < F \leq F_j \leq F_i$:

if $\phi \sim_i F$ then ϕ weakly $\succ_j F$

or

if $F \sim_j \phi$ then F weakly $\succ_i \phi$.

"That is, if the firm with the smaller index is indifferent between the two [fractions] then the firm with the larger index must weakly prefer the smaller [fraction]; reciprocally, if the firm with the larger index is indifferent between the two [fractions], then the firm with the smaller index must weakly prefer the larger [fraction]" (Cave and Salant 1995, 88).

Proof.

Both statements can be verified *mutatis mutandis*.

3) Unconstrained monotonicity.

If firm i is unconstrained and at least one firm is constrained,

$F_N \leq F \leq F_i$

then i's induced profits $\pi_i(F)$ is decreasing in F.

4) Continuity .

π_i (F) is a continuous function.

Proofs.

Unconstrained monotonicity and continuity are proved in Appendix B.

<u>Validity of the median-index theorem in our setting</u>

We have shown a translation of the (generalized) preference assumptions required by Cave and Salant's median-index theorem. This theorem has a crucial element in firms' "ideal points". Therefore, we assume that the set of feasible fractions (of wealth to pay) is a compact collection of non-negative elements. Since π_i (F) is continuous and F lies in a compact interval, each firm i has an ideal point, denoted I_i, such that $\pi_i (I_i) \geq \pi_i$ (F) for all F. Moreover, by unconstrained monotonicity, $I_i \leq F_i$ (cf. Cave and Salant 1995, 89).

Hence Cave and Salant's median-index theorem translates to our setting.

Proof.

Suppose there are N voters (i.e. firms), where N is an odd integer. Denote the median index by m = (N+1)/2. Let I_m be an ideal point of firm m and let F denote any other quota.

If $F < I_m$, voters *1, 2, .., m-1, m* (a majority) would at least weakly prefer I_m. This follows since $F < I_m \leq F_m \leq \min (F_{m-1}, F_{m-2}, \ldots, F_2, F_1)$ and these voters partially agree with *m*.

If instead $F > I_m$, voters *m, m+1,…, N-1, N* (a majority) would at least weakly prefer I_m.

Recall that the cutoffs of these firms are no larger than F_m and that $I_m \leq F_m$. Any *i* such that $F_i \leq I_m$ must weakly prefer I_m to $F > I_m$ (unconstrained monotonicity). As for any *i* such that $I_m < F_i \leq F_m$, such a firm at least weakly prefers I_m to any $F \in (I_m, F_i]$ (since preferences partially agree) and at least weakly prefers F_i to any $F > F_i$ (unconstrained monotonicity). Hence it weakly prefers I_m to any $F > I_m$ (continuity).

We have thus established the existence of at least one Condorcet winner, namely any ideal point (i.e. fraction) of the voter with the median index. That fraction is unique if two additional mild conditions hold (Cave and Salant 1995, 90):

- the firm with the median index has a single ideal point;
- at this ideal point, the preference of every firm unconstrained at I_m is strictly decreasing.[10]

[10] See Cave and Salant (1995) for additional insights.

5.- Summary and final remarks

This study has suggested a simple plausible solution to the problem of collecting aggregate bids for auctions of a discrete good which could be used in non-rival ways by several actors. It is assumed these actors participate in a voluntary game that develops in two stages: in the first stage, group players vote on a common minimum percentage of their wealth to pay; in the second stage, they offer at least the minimum amount voted in the previous stage and therefore collect an aggregate bid. The study has investigated the case where that minimum fraction of wealth is set by majority vote (each player has one vote); a translation of the median-index theorem was applied to our circumstances and the existence of at least one Condorcet winner then established.

Thus, the auctioneer receives bids submitted by group players (for shared use of the good auctioned off) or individual players (for sole use of the good), compares the aggregate bid submitted by group players with the highest bid (if any) submitted by sole bidders, and provisionally assigns the lot accordingly. This continues till there is no excess demand. Then, for instance, if the lot goes to individual exclusive use, the winner pays the larger of the next-highest sole-use bid or the total of the shared-use bids; if the lot goes to shared use, each bidder 'pays' the smallest amount they could have bid without changing the use class (i.e. shared or sole use).

Appendix A

Existence and uniqueness of a pure-strategy Nash equilibrium

Existence

Let $W_N = \sum_1^N w_i$ denote the sum of exogenous wealth of the N players, whose maximization problem is:

$\max \quad b_i\,[f(B) - c_i]$

s.t.

$b_i \geq Fw_i$.

If the constraint is not binding, the F.O.C. requires $f(B) - c_i + b_i\, f'(B) = 0$; therefore, we get $b_i = -\dfrac{f(B) - c_i}{f'(B)}$.

Since $f'(B) < 0$, $b_i > 0$ if $f(B) - c_i > 0$ or, equivalently, $f(B) > c_i$.

If the constraint is binding, firm *i* contributes Fw_i.

Let $\beta_i(B)$ denote firm *i*'s best reply:

$$\beta_i(B) = \max\left\{Fw_i, -\frac{f(B) - c_i}{f'(B)}\right\}$$

for $B \in [FW_N, W_N]$.

Define $\beta(B) = \sum_1^N \beta_i(B)$. Hence $\beta(B)$ is the "aggregate best reply". Since $f(B)$ and $f'(B)$ are continuous and $f'(B) < 0$, $\beta(B)$ is a continuous function. Moreover, if the firm with the lowest cost of capital has positive average net benefit when firms contribute the minimum fraction of their wealth (i.e. $f(FW_N) > c_i$), then the aggregate best-reply contribution is greater than FW_N:

$\beta(FW_N) > FW_N$ as long as $f(FW_N) > c_i$.

Finally $\beta(W_N) \leq W_N$ (the maximum amount of funds that the N firms can contribute is their entire wealth).

It follows that there exists at least one fixed point $B^* \in [FW_N, W_N]$ such that $\beta(B^*) = B^*$.

Assume that total benefit is strictly concave:

$2f'(B) + B f''(B) < 0$ for all $B \in [FW_N, W_N]$; then $2f'(B) + \beta_i(B) f''(B) < 0$ for all $B \in [FW_N, W_N]$ and each firm's second-order condition will be satisfied whenever its first-order conditions hold. Hence, every fixed point of the mapping $\beta(.)$ is a pure-strategy Nash equilibrium.

Uniqueness

We now verify that the left-hand derivative of $\beta(.)$, evaluated at any fixed point B^*, is strictly less than 1 – which implies that there exists a unique fixed point.

If firm *i* is unconstrained,

$$\beta_i(B) = -\frac{f(B) - c_i}{f'(B)}.$$

Hence $$\beta'_i(B) = -\left\{1 + \frac{f''(B)}{f'(B)}\beta_i(B)\right\}.$$

Assume that, as $B \rightarrow B^*$ from the left, *u* firms are unconstrained; summing over the unconstrained firms we obtain:

$$\beta'(B^*)^- = -\left\{u + \frac{f''(B^*)}{f'(B^*)}[\beta(B^*) - FW_{co}]\right\}$$

where FW_{co} is the aggregate contribution of the constrained firms (they must contribute the minimum fraction of their wealth according to F, i.e. Fw_i, which is their best reply).

Since $f'(B^*) < 0$ and $\beta(B) \geq FW_{co}$, $\beta'(B^*)^- \leq 0 < 1$ *provided* $f''(B) \leq 0$.

It remains to show that $\beta'(B^*)^- < 1$ if $f''(B) > 0$.

At any fixed point, $2f'(B^*) + \beta_i(B^*)f''(B^*) < 0$ (since total revenue is strictly concave). Hence, summing over the *u* unconstrained firms

$2uf'(B^*) + [\beta(B^*) - FW_{co}]\, f''(B^*) < 0.$

Adding the negative quantity[11] $2f'(B^*) + \beta(B^*)f''(B^*)$ to the previous inequality (which is negative), we obtain:

$2uf'(B^*) + [\beta(B^*) - FW_{co}]\, f''(B^*) + 2f'(B^*) + \beta(B^*)f''(B^*) < 0$

or, equivalently,

$$2f'(B^*)\,[u + 1] + 2\left\{\beta(B^*) - \frac{FW_{co}}{2}\right\} f''(B^*) < 0\,.$$

Dividing by $- 2f'(B^*) > 0$ we get

$$-[u + 1] - \frac{f''(B^*)}{f'(B^*)}\left\{\beta(B^*) - \frac{FW_{co}}{2}\right\} < 0$$

or, equivalently,

$$-\left\{u + \frac{f''(B^*)}{f'(B^*)}\left[\beta(B^*) - \frac{FW_{co}}{2}\right]\right\} < 1.$$

Since $\dfrac{FW_{co}\; f''(B^*)}{2\, f'(B^*)} < 0$ we obtain

$$-\left\{u + \frac{f''(B^*)}{f'(B^*)}\left[\beta(B^*) - \frac{FW_{co}}{2}\right]\right\} + \frac{FW_{co}\; f''(B^*)}{2\, f'(B^*)} < 1$$

or, equivalently,

$$-\left\{u + \frac{f''(B^*)}{f'(B^*)}\left[\beta(B^*) - FW_{co}\right]\right\} < 1;\ \text{hence}\ \beta'(B^*)^{-} < 1.$$

Appendix B

Unconstrained monotonicity and convexity of the set of fractions binding on firm i

Let B(F) denote the aggregate contribution offered by firms bidding for unlicensed spectrum in the unique Nash equilibrium induced by fraction F (set

[11] Recall total revenue is strictly concave and $\beta(B^*) = B^*$.

by majority-rule vote)[12] and let i be an unconstrained bidder at F. Firm i's profits are:

$\pi_i = \{f\,(B(F)) - c_i\} \cdot b_i\,(B(F))$.

A change in F will affect i's profits:

$$\frac{d\pi_i}{dF} = \frac{dB}{dF}\left\{[f(B(F)) - c_i]\frac{db_i}{dB} + b_i f\,'(B(F))\right\}.$$

For firm i, marginal benefit and cost are equal:

$f(B(F)) + b_i\, f'\,(B(F)) - c_i = 0;$

hence $f(B(F)) - c_i = -\,b_i\, f'\,(B(F))$ and we obtain

$$\frac{d\pi_i}{dF} = \frac{dB}{dF}\left\{[-\,b_i f\,'(B(F)]\frac{db_i}{dB} + b_i f\,'(B(F))\right\} = \frac{dB}{dF}\left\{1 - \frac{db_i}{dB}\right\} b_i f\,'(B(F))$$

where $f\,'(B(F))$ is strictly negative.

Since $b_i(B)$ implicitly solves $f(B) + b_i\, f'\,(B) - c_i = 0$, we can use the implicit function theorem to get that

$$\frac{db_i}{dB} = -\frac{f\,'(B) + b_i f\,''(B)}{f\,'(B)}$$

and, since total benefit is strictly concave, we obtain

$$1 - \frac{db_i}{dB} = \frac{2f\,'(B) + b_i f\,''(B)}{f\,'(B)} > 0\,.$$

Hence $\left\{1 - \frac{db_i}{dB}\right\} b_i f\,'(B(F)) < 0$ and

$$\text{sgn}\,\frac{d\pi_i}{dF} = -\,\text{sgn}\,\frac{dB}{dF}.$$

To show that $\frac{d\pi_i}{dF} \le 0$ as long as some firm is constrained (clearly $\frac{d\pi_i}{dF} = 0$ if no firm is constrained), we verify that $\frac{dB}{dF} > 0$.

[12] Henceforth, to simplify our notation, we will write B(F) without an asterisk.

Let Ω be the set of unconstrained firms and u the number of elements in this set. For each unconstrained firm $i \in \Omega$ we have $f(B) + b_i f'(B) - c_i = 0$. Also, let X be the set of constrained firms and v be the number of its elements ($v = N - u$ and FW_{co} is their aggregate contribution, i.e. $FW_{co} = \sum_1^v Fw_j$, where j is a firm in X). The aggregate contribution collected by the unconstrained bidders is $\sum_1^u b_i = B - FW_{co}$.

Summing over the set of unconstrained firms, we obtain

$$uf(B) + [B - FW_{co}]\, f'(B) - \sum_1^u c_i = 0.$$

Total differentiation gives:

$$\frac{dB}{dF} = \frac{W_{co} f'(B)}{(u+1) f'(B) + (B - FW_{co}) f''(B)}$$

which is zero if no firm is constrained ($W_{co} = \sum_1^v w_j = 0$).

Suppose $W_{co} = \sum_1^v w_j > 0$. Since $f'(B) < 0$ and $\left\{u + \frac{f''(B)}{f'(B)}[B - FW_{co}]\right\} > -1$ (cf. Appendix A), we get

$(u + 1)f'(B) + [B - FW_{co}]\, f''(B) < 0$. Hence $\frac{dB}{dF} > 0$.

Following the reasoning in Cave and Salant (1995), we now use these results to verify that a firm unconstrained at F will remain unconstrained at any looser fraction F_l (where $F_l < F$). For this it is sufficient that the optimal bid of any unconstrained firm i decrease no faster than the minimum contribution required by the voting committee Fw_i, as F decreases:

$db_i \geq dFw_i$ (note that there are both negative); hence

$$\frac{db_i}{dF} = \frac{db_i}{dB}\left(\frac{dB}{dF}\right) \le w_i \,.$$

Since $\frac{db_i}{dB} = -\frac{f'(B) + b_i f''(B)}{f'(B)}$ and $\frac{dB}{dF} = \frac{W_{co} f'(B)}{(u+1)\, f'(B) + (B - FW_{co})\, f''(B)}$, we get that

$$\frac{db_i}{dF} = -\,\frac{W_{co}[b_i f''(B) + f'(B)]}{(u+1)\, f'(B) + (B - FW_{co})\, f''(B)} \le w_i$$

where $u \ge 1$ and $(u + 1)f'(B) + [B - FW_{co}]\, f''(B) < 0$.

If $f''(B) \le 0$ then $W_{co}[b_i f''(B) + f'(B)] < 0$ and $-\frac{W_{co}[b_i f''(B) + f'(B)]}{(u+1)\, f'(B) + (B - FW_{co})\, f''(B)} \le w_i$ clearly holds in this case ($w_i \ge 0$).

Suppose instead that $f''(B) > 0$. Since total benefit is concave, the following inequality holds:

$$B\, f''(B) + 2f'(B) + \left\{\frac{W_{co}}{w_i} + (u-1)\right\} f'(B) < 0.$$

This is equivalent to

$$B\, f''(B) + \left\{\frac{W_{co}}{w_i} + (u+1)\right\} f'(B) < 0$$

which can be manipulated to get

$$f''(B) + \left\{Fw_i W_{co} + w_i (B - FW_{co})\right\} + f'(B)\left\{W_{co} + (u+1) w_i\right\} < 0$$

(note that $w_i B = Fw_i W_{co} + w_i\, [B - FW_{co}]$).

Re-arranging we obtain:

$f''(B)\, Fw_i W_{co} + f''(B)\, w_i\, [B - FW_{co}] + f'(B)\, W_{co} + f'(B)\, (u + 1)\, w_i < 0$

or, equivalently,

$W_{co}\, [f''(B)\, Fw_i + f'(B)] + w_i\, \{f''(B)\, [B - FW_{co}] + f'(B)\, (u + 1)\} < 0.$

Therefore, $- W_{co}\, [f''(B)\, Fw_i + f'(B)] > w_i\, \{f''(B)\, [B - FW_{co}] + f'(B)\, (u + 1)\}$.

Since $\{f''(B)\, [B - FW_{co}] + f'(B)\, (u + 1)\} < 0$ we get

$$-\,\frac{W_{co}[f''(B)\, Fw_i + f'(B)]}{(u+1)\, f'(B) + (B - FW_{co})\, f''(B)} < w_i$$

where $Fw_i \leq b_i$. Hence the following inequality holds:

$$- \frac{W_{co}[b_i f''(B) + f'(B)]}{(u+1) f'(B) + (B - FW_{co}) f''(B)} \leq w_i .$$

This confirms that a firm unconstrained at F will remain unconstrained at any looser quota.

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