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A TOP DOG TALE WITH PREFERENCE RIGIDITIES

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A top dog tale with preference rigidities

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Abstract

With preference rigidities we find Pareto optima of an exchange economy, some of which involve unconsumed endowments. We show that such Pareto Optima can only be attained as market equilibria if there is a top dog in the initial endowment distribution who is richer than the other individuals. The most inegalitarian efficient allocation favouring the top dog is globally stable and it is in the core. For endowment distributions with a top dog, the core contains efficient allocations more equal than the market equilibrium. A voting mechanism or government policy can also offset the top dog's power.

Keywords: Exchange economy; Complements; Top dog allocation.

JEL classification: D50; D61.

1 Introduction

Without externalities, public goods and informational differences, a complete market system can be used to generate efficient outcomes. In an exchange economy, efficiency means both that all goods are consumed and allocational efficiency. Of course, some assumptions are necessary to establish this−typical textbook treatments assume at least local nonsatiation and strict quasiconcavity (Mas-Colell *et al* (1995)). If preferences exhibit heterogeneous complementarities, then there can be rigidities which prevent full utilisation of resources. For example, Marie Antoinette perhaps needed wine and cake and had no use for bread, the populace needed bread and wine but without both each was useless. So, one efficient outcome would involve Marie Antoinette consuming all the cake and wine with the bread being thrown away due to the shortage of wine. And the populace starved. Another efficient outcome would involve Marie Antoinette and the populace sharing the available wine and wasting excess amounts of cake and bread. The point is that there are efficient outcomes

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involving waste, despite some (or all) individuals being nonsatiated in combinations of goods. That is there is "bundled nonsatiation". This surplus of resources is not due to low demand, but occurs because goods are not available in the right combination to be useful. In such a context, an interesting question is then what prices and what types of initial endowment distributions between individuals will decentralise a particular efficient outcome as a market equilibrium. Conversely, what is the role of the initial endowment distribution between individuals in reaching different efficient allocations and, under tatonnement, what are the stability properties of these market equilibria? If Marie Antoinette initially owns most of the wine, the market equilibrium will be likely to lead to the efficient outcome in which the populace starves. But if initial ownership of goods is more equal, then the market allocation will lead to more equal consumption of goods useful to the different groups. We analyse these issues in an exchange economy with cyclical preferences and perfect complements (Scarf 1960, Hirota 1981, Anderson *et al* 2004).

In this scenario, each good enters the preferences of two individuals and each individual gets utility only from two goods, but no pair of individuals care about exactly the same goods. All individuals have perfect complementarity in those goods that they wish to consume. But the goods they desire overlap just partially, e.g. any two of the individuals have something in common but not everything. The original motivation of this setting was to highlight the possibility of global instability of general equilibrium (Scarf 1960) and the theoretical analysis has been confined to that. Here we argue that these environments are important from a normative and empirical point of view. As well as a market exchange economy between consumers, there are other scenarios more rooted in public economics in which preference rigidities matter and other ways in which such allocations can be realised, e.g. voting or bargaining. A natural application of this setting is in social contracting, when the three players are the representative of social groups and the task is to chose how to share a fixed amount of resources with conflicting interests. For example, the allocation could be of environmental or infrastructure variables such as local public services, carbon permits, vouchers, etc. If each social group has a pool of priorities that only partially overlaps with the pool of the other groups and the amount of resources is limited, then the Scarf-like setting emerges. Each representative will then seek their preferred priorities of infrastructure variables within the allocation mechanism used. We show that the initial distribution of bargaining power among representatives and the rule of contracting are decisive for the system outcome.

Previous literature has investigated market equilibrium in this exchange economy with restrictive initial endowment distributions, limiting the attention to stability analysis. In Scarf's seminal paper, it is assumed that each individual has only the total endowment of one of the goods that he wants to consume. Later research contributions are still characterised by special endowment restrictions. For example, Hirota (1981) analyses the market equilibrium assuming that the sum of the initial endowment across goods is equal for each individual and coincides with the aggregate endowment. Anderson *et al* (2004) develop an experimental double auction and allow prices to adjust under a nontatonnement rule, based on the same endowment restrictions as those imposed by Hirota. A consequence of focussing attention just on these endowments is that there is a unique market equilibrium which gives equal utility to all individuals, the prices of all goods are equal and in the equilibrium all goods are fully utilised.

In this paper, we provide a full theoretical analysis of the efficiency, equilibrium and stability properties of these economies with preference rigidities, allowing for more general endowment distributions. Firstly, we characterise the full set of efficient allocations. We show that only three classes of Pareto optima arise. There is a single Pareto optimum in which the efficient allocation exhausts the endowment of all the goods. In all the other efficient allocations, the endowment of one good is totally or partially wasted−we call these *corner allocations*. Scarf's preferences are not strictly convex and also not strictly monotone in all goods. Therefore, the second fundamental theorem of welfare economics cannot be easily invoked. With perfect complementarity, the set of prices which decentralise a particular efficient allocation is generally not unique.We define the set of prices and initial endowment distributions which will decentralise each type of Pareto optimum as a market equilibrium. Moreover, we identify the effects of the initial endowment distribution on the decentralisation of the different allocations. Pareto efficient allocations which imply the total exhaustion of all goods can be decentralised if and only if the individuals have similar endowment distributions of the goods. Pareto efficient allocations without the full utilisations of resources arise in the system if and only if there is a top-dog individual who is in an advantaged position relative to the other second class citizens but who does not initially own the goods exactly in the proportion in which he wishes, and so he trades with the other individuals. Note that in our context the emergence of a top dog is due to a combination of preference rigidities and initial endowment inequalities. The market mechanism is not able to overcome the initial endowment inequality. This source of equilibrium inequality is different from that found in some recent literature. Michele Piccione and Ariel Rubinstein (2007), Allan M. Feldman and Roberto Serrano (2006), for example, give a rationale for the emergence of a top dog in equilibrium by assuming that there is an exogenous ranking scheme on individuals which defines the distribution of power among them, allowing the powerful individual to seize the endowment of the bottom ranked people. In our context, no exogenous ranking scheme is necessary but rigidities in preferences are the principal cause of inequalities in the system.

The set of government interventions necessary to neutralise the top dog and to move from one type of efficient allocation to another in a market system is then discussed. Next for any given initial endowment distribution, we analyse the core and find it is non-empty despite the absence of local nonsatiation. Even with an initial endowment distribution which would generate a top-dog market equilibrium in which the top dog actively trades, the core contains other more equal utility allocations. Similarly, the full egalitarian outcome can result from majority voting. Finally we conduct stability analysis of the market solution under tatonnement.

Specifically, the unique efficient allocation which exhausts the supply of all goods and gives equal "utility" to all individuals can be decentralised using different combinations of prices and endowment sets. Firstly, this efficient allocation can be supported by unequal positive prices for all goods if the initial endowment distributions satisfy a mild set of inequality restrictions. We notice a case in which this allocation is decentralised by good x costing twice as much as good y if and only if the endowment distributions of individuals 1 and 2 satisfy a single restriction. Secondly, we can have equal prices for two of the goods if and only if just two of the individuals have a "similar aggregate" initial endowment distribution. This case tends to exhibit local stability and it is certainly stable if just two individuals have exactly the same initial endowment of each good. Thirdly, it is possible to support the equal utility allocation with three equal prices if and only if the endowment distribution satisfies the restrictions introduced by Hirota; heuristically all three individuals have a "similar aggregate" initial endowment distribution. His restrictions imply that in equilibrium, since prices of all goods are equal, individuals have equal wealth and they can trade goods on a one-for-one basis. This allows individuals to specialise in consumption on the goods they want through trade. The stability properties of this equilibrium have been previously analysed by Scarf and Hirota with a particular endowment distribution. We give stability results for the more general case.

The other efficient allocations in which the endowment distributions are such that a top dog formation emerges imply some waste of resources and can be decentralised only if the price of the good that the top dog does not desire is null. The particular efficient allocation in which the good is totally wasted emerges if and only if one individual is in such a favoured endowment position that he has the total endowments of the goods that he likes and no trade occurs in the market. This extreme allocation is globally stable.

In terms of the core, apart from the Walrasian equilibrium, there is a set of allocations that cannot be blocked by coalitions amongst the three consumers. If the initial endowment distribution leads to a top dog Walrasian equilibrium with trade, there are other core allocations that imply fuller utilisation of resources and more equality than the market equilibrium. Majority rule can completely offset the top-dog position.

The paper is organised as follows. After stating Scarf's preferences, we find the three classes of Pareto optima of this economy. In Section 3, we analyse the feasible types of market equilibria. We next define the set of prices and initial endowment distributions that can decentralise the different Pareto optima (Section 4 and 5). The next sections look briefly at government policies which will move the market economy from an inegalitarian top dog outcome to a more egalitarain outcome, the structure of the core and the contrast between voting and market outcomes with these preferences. We finish with stability analysis of the market equilibria (Section 6).

2 Perfect Complements and Cyclical Preferences

For the sake of simplicity, we consider the original Scarf economy with perfect complements and cyclical preferences of 3 individuals and 3 goods. Individual preferences are given by

$$\begin{aligned} u_1(x_1, y_1, z_1) &= \min\{y_1, z_1\}, \\ u_2(x_2, y_2, z_2) &= \min\{x_2, z_2\}, \\ u_3(x_3, y_3, z_3) &= \min\{x_3, y_3\}. \end{aligned}$$

There is an interlocking set of perfect complementarities in preferences between the three goods.

Note that for convenience we normalise the scale of the economy at 1 unit of each good. Obviously, changing the size of the economy does not affect the nature of the results.

2.1 Pareto Optima

With the strong complementarities, we would expect efficient allocations to involve specialisation in consumption on those goods which individuals wish to consume. For example, allocating any of good 3 to individual 1 yields no Pareto improvement. Moreover, we find that there can be two efficient allocations in which one allocation involves more consumption of one good by one individual (which is of no utility value to him). But both allocations are efficient.

The set of feasible allocations is given by

$$F = \{x, y, z | \Sigma x_h \leq 1, \Sigma x_h \leq 1, \Sigma x_h \leq 1, x \geq 0, y \geq 0, z \geq 0\},$$

where $x = (x_1, x_2, x_3)$, etc. The set of efficient allocations are most easily shown in terms of the efficient utility distributions. Define

$$\begin{aligned} P_1 &= \{x, y, z | (x, y, z) \varepsilon F, u_1(x_1, y_1, z_1) = 1 - a, u_2(x_2, y_2, z_2) = a, u_3(x_3, y_3, z_3) = a, 0 \leq a \leq 1/2\}, \\ P_2 &= \{x, y, z | (x, y, z) \varepsilon F, u_1(x_1, y_1, z_1) = a, u_2(x_2, y_2, z_2) = 1 - a, u_3(x_3, y_3, z_3) = a, 0 \leq a \leq 1/2\}, \\ P_3 &= \{x, y, z | (x, y, z) \varepsilon F, u_1(x_1, y_1, z_1) = a, u_2(x_2, y_2, z_2) = a, u_3(x_3, y_3, z_3) = 1 - a, 0 \leq a \leq 1/2\}. \end{aligned}$$

Thus P_1 is a set of feasible utility distributions which favour individual 1, in the sense that, as a varies, u_1 varies in the interval $(1/2, 1)$, while $u_2 = u_3$ vary in $(0, 1/2)$. In this situation, we refer to the most favoured individual as the top dog. Similarly in P_2, P_3 a different individual is favoured. The set of efficient utility distributions is given by

$$P = P_1 \cup P_2 \cup P_3.$$

The set of efficient allocations is characterised by three types of Pareto optima. Only the first type exhausts the aggregate feasibility constraint. The other cases imply throwing out totally or partially the endowment of one of the goods.

(a) *Class I: no waste.* There is a Pareto optimum in which the individuals get equal utility $u_1 = u_2 = u_3 = 1/2$:

$$
\begin{aligned}
y_1 &= z_1 = 1/2 = 1/2, \\
x_2 &= z_2 = 1/2 = 1/2, \\
x_3 &= y_3 = 1/2 = 1/2,
\end{aligned}
$$

and none of the goods is wasted.

(b) *Class II: the aggregate endowment of one good is partially wasted.* There is an infinite number of other efficient utility distributions which can be reached without consuming the total endowment of one of the goods. For example set $u_1 = u_2 = a, u_3 = 1 - a$. This is attained by consumptions

	x_h	y_h	z_h	u_h
$h = 1$	0	a	a	a
$h = 2$	a	0	a	a
$h = 3$	$1 - a$	$1 - a$	0	$1 - a$
Total	1	1	$2a$	

.

So long as $0 \leq a \leq 1/2$, these allocations are feasible and they cannot be bettered. There is a surplus of good z available, but it cannot usefully be consumed by either individual 3 (he does not want it) nor by individuals 1 and 2 (since there is no matching remaining amount of their complementary good available).

Example 1 *If* $a = 1/4$, *the efficient utility distributions and consumptions are:*

	x_h	y_h	z_h	u_h
$h = 1$	0	$1/4$	$1/4$	$1/4$
$h = 2$	$1/4$	0	$1/4$	$1/4$
$h = 3$	$3/4$	$3/4$	0	$3/4$
Total	1	1	$1/2$	

In such a case, 50% of one of the goods is wasted. Similarly, there are two alternative Pareto optima in which only half of one good is not fully consumed but in which there is a different top dog individual

u_1	u_2	u_3
$1/4$	$3/4$	$1/4$
$3/4$	$1/4$	$1/4$

(c) *Class III: the aggregate endowment of one good is totally wasted.* This class is characterised by three Pareto optima in which one individual gets the total endowment of two of the goods and the third good is just completely wasted:

$$u_1 = 1 \text{ with } y_1 = z_1 = 1; u_2 = u_3 = 0.$$

Here 1 uses all of Y, Z which since these are essential goods for $2, 3$ means that $2, 3$ are restricted to the utility associated with zero consumption of the goods they care about.

(a)-(c) above define the only types of Pareto optima. In any Pareto optimum, two of the goods must be fully allocated for consumption—at most one good may have no useful consumption purpose. If two of the goods were not fully allocated, we could raise the utility of the person who wants those two goods by giving them the lower amount of whatever is leftover, so that worthwhile consumption increases.

In Figure 1, we represent the full set of Pareto optima. The apex shows the Pareto optimum in which all individuals get equal utility. The upper boundary of the pyramid shows the other two classes in which one good is totally or partially wasted.

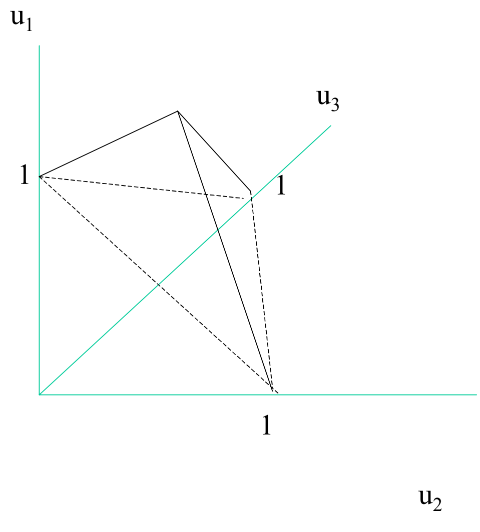

Figure 1: A graphic representation of the different types of Pareto Optima.

3 Market Equilibria

Initial endowments for h are given by $\omega_h = (X_h, Y_h, Z_h)$. Prices are $p = (p_x, p_y, p_z)$. Note also that homogeneity of degree zero in prices implies that we can impose a price normalisation. The two most common are either to set one price equal to unity (but this assumes that any equilibrium will have a positive price in that particular market, i.e. the numeraire good is not in excess supply in equilibrium) or $\Sigma p_i = 1$[1]. Here we use the latter normalisation.

All goods are owned by some individual so that, as the aggregate endowment of each good is unity,

$$\Sigma_h X_h = \Sigma_h Y_h = \Sigma_h Z_h = 1.$$

Demands are given by

$$\begin{aligned} f_{x1} &= 0, f_{y1} = f_{z1} = \frac{p_x X_1 + p_y Y_1 + p_z Z_1}{p_y + p_z}, \\ f_{y2} &= 0, f_{x2} = f_{z2} = \frac{p_x X_2 + p_y Y_2 + p_z Z_2}{p_x + p_z}, \\ f_{z3} &= 0, f_{x3} = f_{y3} = \frac{p_x X_3 + p_y Y_3 + p_z Z_3}{p_x + p_y}. \end{aligned}$$

These are continuous in prices for $p_x, p_y, p_z > 0$, they satisfy the individual budget constraints with equality and they are homogeneous of degree zero in p. Note that they are also continuous at a point at which just one price is zero and the other two prices are positive. However, they are discontinuous at a point at which any two prices are zero.

Since the aggregate endowments of each good are equal to unity, the excess demands reduce to

$$\begin{aligned} E_x &= f_{x2} + f_{x3} - 1, \\ E_y &= f_{y1} + f_{y3} - 1, \\ E_z &= f_{z1} + f_{z2} - 1. \end{aligned}$$

The three excess demand equations are dependent because of Walras Law[2].

For a fixed initial endowment distribution between individuals, an equilibrium is a price vector p, such that there is no aggregate excess demand, and for any good i, if there is excess

[1] By contrast, Scarf used an unusual price normalisation: $\Sigma p_i^2 = 1$ which, combined with the non-negativity of prices, means that prices are restricted to the surface of a non-negative quartersphere.

[2] Walras Law:

supply at p of good i, then $p_i = 0$. That is goods which in equilibrium are in excess supply are priced at zero. Formally, for a given initial endowment distribution between individuals, an equilibrium is a set of prices p_i such that

$$E_i \leq 0, p_i \geq 0, p_i E_i = 0 \quad i = x, y, z.$$

Note that an equilibrium of this economy can never have two prices zero. If, for example, $p_x = p_y = 0$, then individual 3 will have an infinite demand for goods x and y. Since excess demands are continuous (except where two prices are equal to zero) and they satisfy Walras Law, a competitive equilibrium exists (see, for example, Arrow and Hahn (1971)). In this paper, we analyse whether this equilibrium is unique o stable under tatonnement.

In the next section, we find different combinations of endowments and prices (three different prices, proportional prices, two different prices and all equal prices) which decentralise the equal utility Pareto optimum.

4 The Decentralisation of Pareto Optima

4.1 The Equal Utility Pareto Optimum

Here we have $u_h = 1/2$ and all goods are consumed. To represent this as a market equilibrium, there must be an initial endowment distribution and prices such that all excess demands are zero (as each good is fully consumed) and prices are all positive.

From Walras law, we can focus on just two excess demands E_x and E_y. In fact to yield the Pareto optimal allocation, we must have $f_{x2} = f_{y1} = f_{x3} = 1/2$. Again these equations are not all independent, so we focus on the first two $f_{x2} = f_{y1} = 1/2$. This requires

$$p_x X_1 = p_y(1/2 - Y_1) + (1 - p_y - p_x)(1/2 - Z_1), \tag{1}$$

$$\begin{gathered} p_x E_x + p_y E_y + p_z E_z = \\ p_x(f_{x2} + f_{x3} - 1) + p_y(f_{y1} + f_{x3} - 1) + p_z(f_{y1} + f_{x2} - 1) = \\ (p_y + p_z) f_{y1} + (p_x + p_z) f_{x2} + (p_x + p_y) f_{x3} - (p_x + p_y + p_z) = \\ p_x X_1 + p_y Y_1 + p_z Z_1 + p_x X_2 + p_y Y_2 + p_z Z_2 + p_x X_3 + p_y Y_3 + p_z Z_3 - (p_x + p_y + p_z) = 0. \end{gathered}$$

$$p_y Y_2 = p_x(1/2 - X_2) + (1 - p_y - p_x)(1/2 - Z_2). \tag{2}$$

Solving (1),(2) we find the price equilibrium levels:

$$\begin{aligned} p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2 Z_1 - \frac{1}{2}Y_1 + Y_1 Z_2)}{(Y_2 X_1 - Y_2 Z_1 + \frac{1}{2}Y_2 - Z_2 X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1 Z_2 - Y_1 X_2 + Z_1 X_2)} \quad (3) \\ p_y &= \frac{(\frac{1}{2}X_2 - Z_1 X_2 - \frac{1}{2}X_1 + Z_2 X_1 + \frac{1}{2}Z_1 - \frac{1}{4})}{(Y_2 X_1 - Y_2 Z_1 + \frac{1}{2}Y_2 - Z_2 X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1 Z_2 - Y_1 X_2 + Z_1 X_2)}. \end{aligned}$$

The market equilibrium allocation requires just two equations to be satisfied, whilst there are two normalised prices and six free initial endowment variables that can be selected. So there will be an infinity of ways of decentralising the equal utility efficient allocation.

4.2 Supporting the equal utility Pareto Optimum with unequal prices

Here we show that the equal utility Pareto optimum can be decentralised with positive prices iff the endowment of goods for all individuals are collinear, i.e. $(X_h, Y_h Z_h)$ lie in a plane. This condition generalises the endowment restrictions used by Hirota and Scarf. Under their restrictions, decentralisation of this Pareto optimum requires equal prices for all goods whereas under our more general restriction this is not necessary. There are infinitely many collinear endowments and relative price vectors which will do the job. One particularly interesting example arises when the endowment distribution is proportionally distributed among all the individuals. Here the equilibrium prices are unequal between all goods but are in a fixed proportional relationship. Another case is that in which just two goods have equal prices and the endowment of just two individuals is similarly allocated. But if we do require equal prices for all goods, then this Pareto optimum can be attained as a market equilibrium if and only if the individual initial endowments satisfy Hirota's restrictions.

Suppose we take an arbitrary initial endowment distribution ω_1, ω_2 with $X_3 = 1 - X_1 - X_2, Y_3 = 1 - Y_1 - Y_2, Z_3 = 1 - Z_1 - Z_2$:

Proposition 1 *The equal utility Pareto optimum is supported by unequal prices iff*

$$\begin{aligned} \alpha X_1 + \beta Y_1 + \gamma Z_1 &= \kappa, \quad (4) \\ \alpha X_2 + \beta Y_2 + \gamma Z_2 &= \kappa_1, \end{aligned}$$

where α, β are arbitrary constants with $0 < \alpha \neq \beta \neq \gamma < 1$ and $\kappa = (1-\alpha)/2$, $\kappa_1 = (1-\beta)/2$,$\gamma = 1 - \alpha - \beta > 0$.

Proof. See appendix. ■

This endowment distribution leads to a market equilibrium with prices fixed at the value $p_x = \alpha$, $p_y = \beta$. Any pair of endowment distributions with the same value of α and β will generate the same price equilibrium with the same equal equilibrium utility distribution.

Lemma 1 *If (4) holds for some numbers α, β then*

$$\alpha X_3 + \beta Y_3 + \gamma Z_3 = (\alpha + \beta)/2.$$

(4) defines the collinearity restriction between individual endowments. Any pair of endowment distributions which satisfy (4) for given values of α and β will lead to the same equilibrium prices and equilibrium consumptions of each desired good for each individual of 1/2. We can think of the collinearity restriction as imposing limits on the degree of inequality between the initial endowments of different individuals.

Example 2 *The endowment distribution*

$$Z_1 = 0.3; Y_1 = 0.7; X_1 = .04; Z_2 = 0.35; Y_2 = 0.1; X_2 = .59$$

yields $p_x = \alpha = 0.28, p_y = \beta = 0.33$. But the endowment distribution

$$Z_1 = 0.3; Y_1 = 0.4; X_1 = .4; Z_2 = 0.35; Y_2 = 0.1; X_2 = .59$$

yields exactly the same equilibrium prices and utility distribution.

We can use (4) to generate special cases of endowment distributions in which the equilibrium prices have special properties. For example, one interesting case might be that in which in equilibrium good y is twice as expensive as good x. In such a case, the equal utility distribution can be supported by p_y costing twice p_x if and only if in (4) $\beta = 2\alpha$.

Example 3 *If we choose $a = 1/6$, then $p_x = 1/6$; $p_y = 1/3$ and $p_z = 1/2$ and $X_1 = 2.5 - 2Y_1 - 3Z_1$, and $X_2 = 2 - 2Y_2 - 3Z_2$. Then to be sure that $X_1, X_2 \geq 0$, we need $2Y_1 + 3Z_1 \leq 2.5$ and $2Y_2 + 3Z_2 \leq 2$. This implies $Y_1 \leq 1.25 - 1.5Z_1$, and $Y_2 \leq 1 - 1.5Z_2$.*

This gives us a four parameter family of endowment distributions. But we need $Y_1, Y_2 \geq 0$, *which means we must have* $Z_1 \leq 1.25/1.5$ *and* $Z_2 \leq 1/1.5$. *Suppose we fix* $Z_1 = Z_2 = 1/3$. *Then we can choose any values* $Y_1 \leq .75$ *and* $Y_2 \leq .5$ *which make* $X_1 = 1.5 - 2Y_1 \leq 1$ *(for example,* $Y_1 \geq .25$*) and* $X_2 = 1 - 2Y_2 \leq 1$ *(for example,* $Y_2 \geq 0$*). The upshot is an infinite number of endowments satisfying* $.25 \leq Y_1 \leq .75$ *and* $0 \leq Y_2 \leq .5$, *all of which will give equilibrium with* $p_y = 2p_x$ *and equal utility for the individuals. Obviously, this could be replicated for any* $0 < a < 1/3$ *and for other values of* Z_1, Z_2.

Another case of some interest is that in which in equilibrium goods x and y are equally expensive. Then two individuals trading these goods between themselves would be in a similar position of relative advantage. The equal utility Pareto optimum is supported by $p_x = p_y \neq p_z$ for all goods iff in (4) $\beta = \alpha$. This condition is certainly satisfied if individuals 1 and 2 have exactly the same amount of each good ($X_1 = X_2$; $Y_1 = Y_2$; $Z_1 = Z_2$). More generally, it is also satisfied when the sum of the endowments of two goods of individual 1 and 2 is equal and also they have identical endowments of the third good in the sense that

$$X_1 + Y_1 + Z_1 = X_2 + Y_2 + Z_2 = k \text{ and } Z_1 = Z_2. \tag{5}$$

When $\beta = \alpha$ in (4), the equilibrium prices (13) are

$$\begin{aligned} P &= p_x = p_y = \frac{Z_3}{(2Z_3 + 1 - X_3 - Y_3)} = \frac{Z_3}{k + 3Z_3}, \\ p_z &= 1 - 2p = \frac{k + Z_3}{k + 3Z_3}, \end{aligned} \tag{6}$$

where $k = 1 - X_3 - Y_3 - Z_3$. This gives a whole family of values of the initial endowment distributions all of which generate positive prices $p_y = p_x \neq p_z$ and which generate the market equilibrium giving equal utility for all individuals. Figure 2 plots these alternative equilibrium prices as a function of $k = 1 - X_3 - Y_3 - Z_3$ and Z_3.

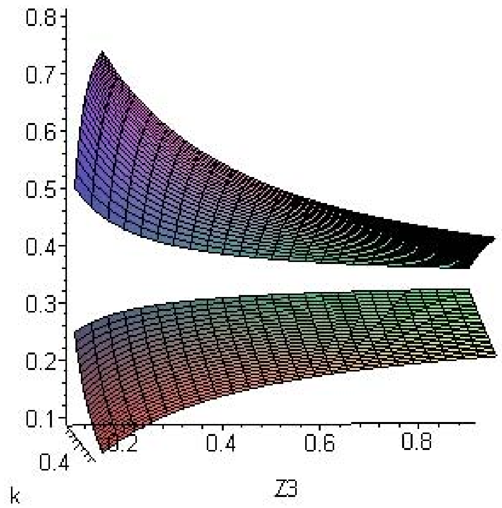

Figure 2: Equilibrium prices P, p_z as a function of k and Z_3.

For example, if $k = 0.2, Z_3 = 0.25$ then $p_x = .263, p_z = .474$. And so on for other combinations.

4.3 Supporting the equal utility Pareto Optimum with equal positive prices

Scarf and Hirota use particular distributions of initial endowments and show that with these, $p_x = p_y = p_z = 1/3$ gives an equilibrium with equal utilities of 1/2. Hirota's class is defined by

$$X_h + Y_h + Z_h = 1 \quad \text{for all } h.$$

In fact Scarf's endowments, i.e. $Y_1 = Z_2 = X_3 = 1$ and all others zero, are a special case of Hirota's class of endowments. Hirota's endowments have the strong interpretation that, when they hold, all individuals have equal wealth if prices are equal for all goods. We can derive this class of endowments from (4) by setting $\alpha = \beta = 1/3$

$$\begin{aligned} X_1 + Y_1 + Z_1 &= 1, \\ X_2 + Y_2 + Z_2 &= 1. \end{aligned}$$

We can then ask what is the full set of initial endowment distributions which make $p_x = p_y = p_z = 1/3$ a market equilibrium and which leads to the equal utility Pareto optimum.

Proposition 2 *The equal utility Pareto optimum is supported by equal positive prices for all goods iff the Hirota conditions hold.*

Thus we have shown that a market equilibrium with equal prices for all three goods supports the equal utility Pareto optimum iff the initial endowments satisfy the Hirota endowment conditions. The equilibrium with equal quantity and prices is obtained when the total endowment, $X + Y + Z = 3$, is equally distributed among individuals. On average, every individual has the same power in contracting since every individual has got a third of the total initial endowment. Each individual endowment is $X_h + Y_h + Z_h = 1$ for each $h = 1, ..., 3$). Setting the prices equal allows one unit of any good to exchange for one unit of any other good so eg individual 1 can sell say 1/3 of a unit of X (which he does not want) and buy 1/6 of a unit of each of Y, Z which he does want.

5 Decentralisation of Corner Pareto Optima

By definition, in a corner Pareto optima one individual has higher utility than the other two who have equal utility. We refer to the individual who is better off in this Pareto optimum as the top dog. Markets can ensure that this utility distribution is reached by finding prices and a suitable endowment to ensure that the top dog has higher equilibrium wealth than the other individuals. Below we characterise the prices and the exact endowment distribution restriction for each type of corner Pareto optimum. One aspect of the endowment restriction is that the top dog must have a sufficiently large endowment of at least one of the goods which he wishes to consume.

5.1 Corner Pareto Optima Class II

Pareto optima Class II have the form $u_h = 1 - a, u_k = a = u_l$ for $h, k, l = 1, 2, 3$. If we analyse one case say $u_1 = 1 - a, u_2 = a = u_3$ the others will follow.

In this case we know that $y_1 = z_1 = 1 - a; x_2 = z_2 = a; x_3 = y_3 = a$ with other consumptions being zero. Generally, we think of 1 as being the favoured individual so that $a < 1/2$, in which case less than the total endowment of x is consumed at the Pareto optimum. In market terms, prices must be such that x is in excess supply. To decentralise this class of Pareto Optima as a market equilibrium, it must be that $p_x = 0$. We know that the total endowment of goods y and z is consumed, so in market equilibrium they must exhibit zero excess demand. So we can take $p_y, p_z > 0$ and for example normalise the prices so that $p_x + p_y + p_z = p_y + p_z = 1$. It follows that $p_z = 1 - p_y$. This leaves p_y as the only price to be determined, and we have two equations that must hold: the demand for goods consumed by individual 1 must equal $1 - a$ and those by individuals 2 must equal a. (It follows by Walras law that also the demand for individual 3 must equal a.)

Individual 1 wants to sell good x and buy $1 - a$ units of good z and y. But good x does not have any value ($p_x = 0$). The net-trade condition equivalent to his demands is

$$0 = p_y((1-a) - Y_1) + (1-p_y)((1-a) - Z_1). \tag{7}$$

Note that if $Y_1 = Z_1$ the individual will not trade at all but will just consume his initial endowment. Also 1 must own initially at least $1 - a$ of one of the goods that he wishes to consume.

Turning to the other individuals, individual 2 wants to sell good y and buy good x and z

$$p_y Y_2 = (1-p_y)(a - Z_2),$$

while individual 3 wants to sell good z and buy good x and y:

$$(1-p_y)Z_3 = p_y(a - Y_3).$$

Proposition 3 *Pareto optima with utility distributions $u_1 = 1-a, u_2 = u_3 = a$ is supported with prices $p_x = 0, 0 < p_y = k \neq p_z < 1$ iff*

$$\begin{aligned} kY_1 + (1-k)Z_1 &= 1-a, \\ kY_2 + (1-k)Z_2 &= (1-k)a, \end{aligned} \tag{8}$$

with $k \neq (1-k)$.

In this case, we have $p_y = k$ and $p_z = 1 - k$. Note that the conditions in the proposition are like the linear endowment restrictions (4) but involving only two goods (i.e. y and z). Of course this is because the distribution of x is immaterial since its price is zero.

To support the corner Pareto optima, what matters is the endowment/wealth distribution. In the examples above, individual 1 is like a top dog with most of the endowment. The wealth of individuals 2 and 3 valued at the equilibrium prices is lower than the wealth of individual 1 valued at the equilibrium prices, since $a \leq 1/2$ and $0 < k < 1$. Note that although the bottom dogs 2 and 3 have equal equilibrium utility, in general their wealths valued at equilibrium prices differ. If $k = 1/2$ they have equal wealth, but if $p_y = k < 1/2$ (and so $p_z > 1/2$), individual 3 who wants to consume x and y has lower wealth than individual 2, who wants to consume x and z.

A special case of (8) is of particular interest. Suppose that we select the endowments so that

$$Y_2 + Z_2 = Y_3 + Z_3 = a \Rightarrow Y_1 + Z_1 = 2(1 - a),$$

then the ratio of endowments in (8) are equal to unity. But since these common ratios are equal to the price ratio between goods y and z, this then means that we can take $p_y = k = 1/2 = p_z$ so that the two goods that have positive value in equilibrium are equally valued. Then this initial endowment distribution gives equal wealth to individuals 2 and 3 when valued at the equilibrium prices, although each has lower wealth than the top dog individual 1.

Proposition 4 *Let $a \leq 1/2$. Pareto optima with utility distributions $u_1 = 1 - a, u_2 = u_3 = a$ is supported with prices $p_x = 0, p_y = p_z = 1/2$ iff*

$$Y_1 + Z_1 = 2(1 - a), Y_2 + Z_2 = Y_3 + Z_3 = a.$$

In equilibrium individuals 2 and 3 are equally wealthy but both are clearly less wealthy than individual 1.For example, setting $a = 1/4; k = .5; Z2 = 1/6$ gives $Y_2 = .167, Y_3 = .183, Z_3 = .033$. The point is that for any a, there is an infinity of positive but unequal prices $p_y \neq p_z$ with associated individual initial endowment distributions which lead to the market equilibrium with $u_1 = a, u_2 = u_3 = 1 - a$.

5.2 Corner Pareto Optima Class III

In this class, the Pareto optimum displays extreme inequity: $u_1 = 1, u_2 = u_3 = 0$. This can be supported as a market equilibrium only if individual 1 has got all the endowment of the two goods that he likes, whatever the distribution of the good that he does not want among the other individuals. The net-trade conditions in this case for individual 2 and 3 are respectively $kY_2+(1-k)Z_2 = 0$ and $kY_3+(1-k)Z_3 = 0$, which implies that $Y_2 = Z_2 = Y_3 = 0$ (since $0 < k < 1$ and $Y_h \geq 0, Z_h \geq 0$) and so from the aggregate endowment availability: $Y_1 = Z_1 = 1$.

For example, if $X_1 = 0.3, Y_1 = 1, Z_1 = 1, X_2 = 0.5, X_3 = 0.2; Y_2 = Z_2 = Y_3 = Z_3 = 0$ individual 1 has the total endowment of the two goods y and z that he wishes to consume. Then $u_1 = 1$, $u_2 = u_3 = 0$ and no trade occurs. Each individual just keeps his original endowment although for both individuals 2 and 3 they have no use for one of the goods with which they may be endowed. The prices p_y,p_z are then irrelevant and they can be set at arbitrary levels within the price normalisation.

Proposition 5 *The Pareto optimum with utility distribution* $u_1 = 1, u_2 = u_3 = 0$ *is supported with prices* $p_x = 0$,*and* $p_y > 0$, $p_z > 0$ *iff*

$$Y_1 = Z_1 = 1, Y_2 = Z_2 = Y_3 = Z_3 = 0.$$

In this case the top dog interpretation is extremely inequitable: individuals 2 and 3 have only the endowment of good x, that has no value, while individual 1 has wealth 1 again valued at any prices.

6 Policy to Move Between Equilibria

We have characterised endowment distributions which will lead to market equilibrium prices and consumptions which are Pareto efficient but have very different equity properties. Faced with an endowment distribution leading to a top dog outcome, an egalitarian government may wish to use either direct commodity transfers or, failing that, fiscal policy to move to the equal utility efficient outcome.

Suppose that the initial endowment distribution is such that individual 1 is a top dog in the Walrasian equilibrium and so satisfies:

$$\begin{aligned}
\frac{k}{1-a}Y_1 + \frac{(1-k)}{1-a}Z_1 &= 1, a < 1/2, 0 < k < 1 \\
\frac{k}{(1-k)a}Y_2 + \frac{1}{a}Z_2 &= 1,
\end{aligned} \tag{9}$$

We showed in the previous section that to have an egalitarian solution in the general case of unequal prices, the endowment distribution ($X_i^{'}$, $Y_i^{'}$, $Z_i^{'}$) should satisfy:

$$\begin{aligned}
\frac{2\alpha}{(1-\alpha)}X_1^{'} + \frac{2\beta}{(1-\alpha)}Y_1^{'} + \frac{2(1-\alpha-\beta)}{(1-\alpha)}Z_1^{'} &= 1, \\
\frac{2\alpha}{(1-\beta)}X_2^{'} + \frac{2\beta}{(1-\beta)}Y_2^{'} + \frac{2(1-\alpha-\beta)}{(1-\beta)}Z_2^{'} &= 1
\end{aligned} \tag{10}$$

for arbitrary $0 < \alpha \neq \beta < 1$. What the government could do if it has complete commodity transfer power is to set $X_1^{'} = X_2^{'} = 0$, which is a very simple rule: the potential top dog has to be neutralised by confiscating the good that he wishes to sell. In addition we need $\alpha, \beta, Y_i^{'}, Z_i^{'}$ $i = 1, 2$ such that

$$\begin{aligned}
\frac{2\beta}{(1-\alpha)}Y_1^{'} &= \frac{k}{1-a}Y_1, \frac{2(1-\alpha-\beta)}{(1-\alpha)}Z_1^{'} = \frac{(1-k)}{1-a}Z_1 \\
\frac{2\beta}{(1-\beta)}Y_2^{'} &= \frac{k}{(1-k)a}Y_2, \frac{2(1-\alpha-\beta)}{(1-\beta)}Z_2^{'} = \frac{1}{a}Z_2
\end{aligned}$$

which can be achieved for arbitrary α, β by suitable choice of $Y_i^{'}, Z_i^{'}$.

Of course, if the government has the power to redistribute goods, it can also just redistribute directly to the equal utility allocation. This will then result in a no trade market equilibrium. More interestingly, the government may not have direct redistribution power but does have commodity taxation power. For example, it can impose ad valorem taxes on buyers so that the buyer pays $p_i^{'} = p_i + t_i$, while the seller just receives p_i.

In such a context, demands are given by:

$$\begin{aligned}
f'_{x1} &= 0, f'_{y1} = f'_{z1} = \frac{p_x X_1 + p_y Y_1 + p_z Z_1}{p_y + p_z + t_y + t_z}, \\
f'_{y2} &= 0, f'_{x2} = f'_{z2} = \frac{p_x X_2 + p_y Y_2 + p_z Z_2}{p_x + p_z + t_x + t_z}, \\
f'_{z3} &= 0, f'_{x3} = f'_{y3} = \frac{p_x X_3 + p_y Y_3 + p_z Z_3}{p_x + p_y + t_x + t_y}.
\end{aligned}$$

Zero tax revenue for the government requires

$$t_x\left(f'_{x2}+f'_{x3}\right)+t_y\left(f'_{y1}+f'_{y3}\right)+t_z\left(f'_{z1}+f'_{z2}\right)=0.$$

Using the price normalisation, to ensure the equal utility outcome we need

$$\begin{aligned}\frac{p_x(X_2-Z_2)+p_y(Y_2-Z_2)+Z_2}{1-p_y+t_x+t_z} &= 1/2\\ \frac{p_x(X_3-Z_3)+p_y(Y_3-Z_3)+Z_3}{p_x+p_y+t_x+t_y} &= 1/2.\end{aligned}$$

The zero tax revenue constraint together with these last two equations gives us three equations from which we can solve for the three tax rates.

Obviously, it is possible to fix a set of transfers and taxes to move from an egalitarian to a top dog solution using the same methodology.

7 The Core

There are some robust properties of the core:

(i) any allocation in the core is Pareto optimal

(ii) any allocation in the core must give each individual a utility level at least as great as that achieved by consuming their initial endowment (individual rationality)

(iii) so long as preferences are at least locally nonsatiated, any competitive equilibrium is in the core.

In the Scarf economy, we do not have local nonsatiation of preferences: increases in the availability of just one good to an individual will not necessarily increase their utility. Nevertheless properties (i) and (ii) above obviously hold in the Scarf economy.

More interesting, property (iii) also holds in the Scarf economy. This property essentially holds so long as a weak utility increase for all members of a coalition will increase the aggregate cost of the new consumption allocation with goods valued at equilibrium prices. I.e., if (x^*, y^*, z^*) is a market equilibrium for the given endowment distribution, supported by prices p, then for a sub-allocation (x, y, z) to a coalition S, if it is true that $u_h(x_h, y_h, z_h) \geq u_h(x_h^*, y_h^*, z_h^*)$ for all coalition members (with strict inequality for at least one member), it means that

$$p_x\sum_{h\in S}x_h+p_y\sum_{h\in S}y_h+p_z\sum_{h\in S}z_h>p_x\sum_{h\in S}x_h^*+p_y\sum_{h\in S}y_h^*+p_z\sum_{h\in S}z_h^*.$$

The coalition cannot block x^*, y^*, z^* by x, y, z. In the Scarf economy, in equilibrium any good which is in excess supply has a zero price and that good is then of no value to the coalition members. So utility increases for coalition members must involve increased consumption of good(s) with a positive price. Hence, the aggregate cost to the coalition of a utility superior allocation must be greater than the cost of the equilibrium allocation to the coalition.

In addition to (i)-(iii), for a given initial endowment distribution, only the subset of feasible Pareto efficient allocations that is not blocked by any of the possible two individual coalitions is in the core. It is easy to show that the condition for an allocation to be unblocked by two person coalitions is that the sum of the initial individual endowments of the good which is commonly desired by the two consumers should be less than the amount of that good available in the allocation. Individual 1 wants to consume y and z in equal proportions, individual 2 good x and z, while individual 3 good x and y in equal proportions. An individually rational Pareto efficient allocation (x, y, z) is in the core and it is unblocked by any two individual coalitions if $Y_1+Y_3 < y_1+y_3$, $Z_1+Z_2 < z_1+z_2$ and $X_2+X_3 < x_2+x_3$.

For example, consider individuals 1 and 2. If $Z_{12} < a$, then neither the single or two person coalition can block the efficient allocation $u_1 = u_2 = a$, since also $Z_1, Z_2 \leq Z_{12}$. But if $X_{12}, Y_{12}, Z_{12} > a$, the two person coalition can block this efficient allocation. Similarly, if $Y_1, Z_1 > a$ or $X_2, Z_2 > a$, a single person can block this efficient allocation.

Suppose that the initial endowment distribution leads to a market equilibrium in which 1 is the top dog, so that $p_x = 0$ and $u_1 = 1 - a, u_2 = u_3 = a, a < 1/2$. Consider the efficient utility distribution $u_1 = 1 - b, u_2 = u_3 = b, a < b < 1/2$. This is in the core (i) since it is Pareto optimal. (ii) it cannot be blocked by any two person coalition since $Y_1 + Y_3 \leq 1 = 1 - b + b$, similarly $Z_1 + Z_2 \leq 1$. It remains to show that is individually rational. The endowment distribution satisfies $kY_1 + (1-k)Z_1 = 1 - a$ and, if the top dog actively trades in the market equilibrium, then $Y_1 \neq Z_1$. Suppose $Y_1 < Z_1$, the most that 1 can achieve from autarky is then $u_1 = Y_1 < 1 - a$. Hence, if we choose b so that $Y_1 < 1 - b$, then 1 cannot block the utility distribution $1 - b, b, b$. Individuals 2 and 3 also cannot block the b allocation. They cannot individually block the market allocation with utility a but, since $b > a$, a fortiori they cannot block they allocation giving them utility b.

8 Application to social choice

In a social choice context, the three players are the representatives of social groups. The task is to choose how to share a given amount of resources when there are conflicting interests. Indeed, each representative has interests to promote a particular "package" of political interventions that only partially overlaps with the "package" desired by the other candidates. Thus the individual bargaining power is crucial for the selection of the set of social alternatives that will finally emerge. For example, we could think of three regional jurisdictions each of whom has an initial endowment of three different local public goods, e.g. nurses in health delivery, teachers in education or soldiers in defence. Summing the endowments over the jurisdictions gives national endowments for health, education and defence. With a market system, the equilibrium wage rates for different public sector workers and the mix of service provision in the different jurisdictions will emerge. If the resource endowments are such that there is a top dog Walrasian equilibrium, then this jurisdiction will have plentiful provision at the expense of the other two. Generally, this type of equilibrium will involve public sector unemployment of one category of worker and displays inequality. If the allocation is by simple majority voting, the relatively worse off jurisdictions can enforce the equal provision outcome for those services that they value in each jurisdiction. For example, consider the allocation process in which starting from the initial endowment distribution, individuals take turns to propose a new feasible allocation as an alternative to the status quo. If two individuals at least vote in favour, the proposal becomes the new status quo. Then the next individual can propose a new allocation which is voted on against the current status quo. The final allocation if it exists is one which cannot be defeated in majority vote against any new proposal by any individual (Borck 2007). In this context, this coincides with the equal utility distribution. Suppose that the initial endowment distribution is such that the unique market equilibrium utilities generate, say, 3 as a top dog. Then either 1 or 2 can propose the equal utility allocation which will not be defeated in majority vote. And there is no alternative feasible allocation which 3 can propose which will overturn the equal utility outcome in majority vote. Any such allocation must give at least two individuals $u_h > 1/2$, which would require more than one unit of the good which they commonly value. And so cannot be feasible. One region, one vote neutralises the economic power of the resource rich region.

9 Stability Of Market Equilibria Under Tatonnement

The original interest in the economy put forward by Scarf was in the stability properties of the equal price equilibrium under a tatonnement rule for price adjustment. Scarf showed that with his particular initial endowment distribution the unique market equilibrium $p_x = p_y = p_z = 1/3$ corresponding to the Pareto optimum with equal utilities was globally unstable under the price normalisation that he used. Hirota showed that other initial endowment distributions also lead to the equal price equilibrium and that for these other distributions (within the Hirota class but excluding the Scarf case) there was a tendency to local and global stability.

In general, for local stability, the excess demand functions must be downward sloping in their own price and the feedback cross effects between markets should be "small" in comparison with the own price effects. Generally, we can write the Jacobian of the excess demand functions for x and y as

$$J = \begin{bmatrix} \partial E_x/\partial p_x & \partial E_x/\partial p_y \\ \partial E_y/\partial p_x & \partial E_y/\partial p_y \end{bmatrix}, \tag{11}$$

so that

$$det(J) = \partial E_x/\partial p_x \partial E_y/\partial p_y - \partial E_x/\partial p_y \partial E_y/\partial p_x,$$

and $trace(J) = \partial E_x/\partial p_x + \partial E_y/\partial p_y$. If the excess demand functions are downward sloping in their own price then the trace is always negative. The condition for the determinant to be positive (and hence for two eigenvalues whose real parts are negative and local stability) is that

$$\partial E_x/\partial p_x \partial E_y/\partial p_y > \partial E_x/\partial p_y \partial E_y/\partial p_x.$$

We can think of this as saying that the aggregate of cross market effects (the LHS) should be small in absolute value relative to the own price effects.

9.1 Stability of Equilibrium with Equal Utility

9.1.1 The General Case

To explore local stability with an arbitrary initial endowment distribution satisfying (4), we can linearise the excess demand functions around the equilibrium prices $p_x = \alpha, p_y = \beta$ and compute the trace and the determinant (see Appendix B). The condition for the determinant to be positive is quite simple:

$$(1 - 2Z_2)(1 - 2Y_1) + 2Y_2(1 - 2Z_1) > 0$$

but the trace is more complicated.

In section 4.2, we have shown that in general there are alternative initial endowment distributions which generate equilibrium with the same unequal prices. Some of these endowment distributions which yield the equal utility equilibrium outcome are locally stable, others are locally unstable even though the equilibrium prices are the same (see Appendix B).

Example 4 *If* $Z_1 = 0.3$; $Y_1 = 0.7$; $Z_2 = 0.35$; $Y_2 = 0.1$ *and we take* $X_1 = .043$, $X_2 = .591$, *then* $p_x = \alpha = 0.28$ *and* $p_y = \beta = 0.33$. *With these values, the determinant of the Jacobian in a neighbourhood of equilibrium is* $-.095$ *and the trace is* $-.943$. *The equilibrium is locally unstable since the determinant is negative-locally it is a saddlepoint. On the other hand, if we take* $Z_1 = 0.3; Y_1 = 0.4$; $Z_2 = 0.35$; $Y_2 = 0.1$; $X_1 = .396$ *and* $X_2 = .591$, *then again* $p_x = \alpha = 0.28, p_y = \beta = 0.33$ *but now the determinant has a value of* $.331$, *while the trace is equal to* -1.448. *In this case the equilibrium is locally stable.*

In the two examples, we have given what matters a lot is the relative ownership by individual 1 of goods y and z. This is interesting since they are both goods he wishes to consume.

In the case in which the equal utility Pareto optimum is decentralised by either $p_x = 2p_y$ or $p_x = p_y$, local stability of the walrasian equilibrium is similarly ambiguous. Details and examples are in the Appendix. There are some special endowment restrictions which ensure local stability of the equilibrium with two equal prices, e.g. two individuals have equal aggregate endowments of two of the goods and an identical amount of the third good such as $X_1 + Y_1 = X_2 + Y_2$ and $Z_1 = Z_2$.

9.1.2 Stability of Equilibrium with Equal Prices and Equal Utilities

When the endowment distribution satisfies the Hirota restrictions we need a single endowment restriction to ensure that the equal price, equal utility equilibrium is locally stable. The endowments must satisfy

$$X_2 > \max\{1 - 2Y_1 - Y_2 - 2X_1, \frac{(2Y_1 - 1/2)}{2X_1 + 1 - 2Y_1}\},$$

which combines a condition which Hirota initially found (under a different price normalisation for the determinant to be positive) with a condition for the trace to be negative.

9.2 Stability of Equilibrium with Unequal Utilities

Pareto optima class III are globally stable. If one individual is in such a favoured position that he has the total endowment of the two goods he wishes to consume and no-trade occurs in the market, then the equilibrium displays strong stability properties. In such a case, the other two bottom class citizens are permanently rationed to their initial endowments without achieving any utility. In the other corner Pareto optimum class, still there is a top dog citizen but the difference in terms of wealth with regard to the other citizens is not so remarkable as in the no trade case. The equilibrium that emerges in this case is stable for any initial conditions starting with a zero price for the good which is in excess supply (see Appendix C). However starting with arbitrary initial conditions, we show that for equilibria with some trade which have $p_x = 0$ and individual 1 as the top dog, the sign of the determinant and the trace are ambiguous (see Appendix C.2).

10 Conclusion

Generally in exchange economies with nonsatiated preferences, Pareto efficiency requires the aggregate endowments of each good to be fully consumed. We analyse the case in which, although there is bundled non satiation, Pareto efficiency occurs without the full utilisation of resources. The set of bundled nonsatiated goods that each individual wishes to consume overlaps just partially.

Due to the symmetry of the economy, the obvious Pareto optimum is equal utility for all individuals. In a market economy, we show that this is the equilibrium if individuals have similar initial equal opportunities. But there are many other efficient allocations in which there is a single top dog (Marie Antoinette) and the other two individuals are second class citizens (the populace).

We show that markets are actually quite flexible in this setting. All types of efficient allocations can be reached as market equilibria for suitable initial endowment distributions. We find the endowment distributions and market equilibrium prices which will decentralise the different Pareto optimum configurations, even if the nonsatiation conditions of the fundamental theorem of welfare economics are not strictly satisfied.

When a top dog allocation arises in the market, Pareto efficiency is reached without full exploitation of all the resources. This occurs if in the system there is an individual that is in a favourable endowment position owning the majority of the aggregate endowment of the goods that she wishes to consume but not exactly in the correct proportion and she trades. After the market trade, she still maintains her privileged position and any coalition between the second class citizens cannot overcome this outcome. The market allocation with this endowment distribution is in the core. With these rigidities in preferences, prices and market trade cannot overcome the basic inequality in the endowment distribution. We show that, when there is a top dog in this sense, the market equilibrium supporting the unequal Pareto optimum has strong stability properties. However, the source of a top dog is essentially in the initial endowment distribution. In this sense, markets cannot serve to offset initial inequalities. Perhaps one consolation is that any individual who is rationed out of markets by prices at least has some company. However, government policy can correct the initial inequality either through direct resource transfer or through commodity based taxation. An other way of overcoming the differential in economic power with such preference rigidities is to allocate consumption bundles by majority voting or through cooperative mechanisms.

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A Appendix

Proof of Lemma 1

Suppose that:

$$\begin{aligned}\alpha X_1 + \beta Y_1 + (1-\alpha-\beta) Z_1 &= (1-\alpha)/2, \\ \alpha X_2 + \beta Y_2 + (1-\alpha-\beta) Z_2 &= (1-\beta)/2.\end{aligned}$$

Summing these we obtain

$$\alpha(X_1+X_2) + \beta(Y_1+Y_2) + (1-\alpha-\beta)(Z_1+Z_2) = 1-\alpha/2-\beta/2.$$

But there is an aggregate endowment of unity of each good so this implies:

$$\alpha(1-X_3) + \beta(1-Y_3) + (1-\alpha-\beta)(1-Z_3) = 1-\alpha/2-\beta/2,$$

and rearranging this we derive:

$$\alpha X_3 + \beta Y_3 + (1-\alpha-\beta)Z_3 = \alpha/2 + \beta/2.$$

■

Proof of Proposition 2 (Case with different prices)

a) Suppose that the price are unequal and such that: $p_x = \alpha$, $p_y = \beta$ and $(1-\alpha-\beta) = p_z$, with $0 < 1-\alpha-\beta < 1$, and $0 < \alpha \neq \beta \neq \gamma < 1$. The equilibrium conditions become:

$$\begin{aligned} f_{y1} &= \frac{\alpha X_1 + \beta Y_1 + (1-\alpha-\beta)Z_1}{(1-\alpha)} = 1/2, \\ f_{x2} &= \frac{\alpha X_2 + \beta Y_2 + (1-\alpha-\beta)Z_2}{(1-\beta)} = 1/2, \end{aligned} \tag{12}$$

which imply:

$$\begin{aligned} \alpha X_1 + \beta Y_1 + \gamma Z_1 &= \kappa, \\ \alpha X_2 + \beta Y_2 + \gamma Z_2 &= \kappa_1, \end{aligned}$$

where $\kappa = 1/2(1-\alpha)$, $\kappa_1 = 1/2(1-\beta)$ and $\gamma = 1-\alpha-\beta > 0$.

(b) Conversely suppose the conditions (4) hold. Then we have to show that this implies that $p_x = \alpha$; $p_y = \beta$. Again multiplying through (12), we get the linear system:

$$\begin{aligned} p_x X_1 + p_y Y_1 + (1-p_x-p_y)Z_1 &= (1-p_x)/2, \\ p_x X_2 + p_y Y_2 + (1-p_x-p_y)Z_2 &= (1-p_y)/2. \end{aligned}$$

Solving these linear equations we get:

$$\begin{aligned} p_y &= -\frac{(-\frac{1}{2}X_2 + Z_1X_2 + \frac{1}{2}X_1 - Z_2X_1 - \frac{1}{2}Z_1 + \frac{1}{4})}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)}, \\ p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2Z_1 - \frac{1}{2}Y_1 + Y_1Z_2)}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)}. \end{aligned} \tag{13}$$

This solution requires that the determinant condition

$$(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2) \neq 0$$

should hold.

Recalling the general Hirota conditions (4):

$$X_1 = (\frac{-\beta Y_1 - \gamma Z_1 + \kappa}{\alpha}),$$

$$X_2 = (\frac{-\beta Y_2 - \gamma Z_2 + \kappa_1}{\alpha}),$$

and substituting them in (13) gives $p_x = \alpha; p_y = \beta$. ■

The sufficient and necessary conditions to decentralise the other special cases of of the equal utility distribution with i) p_y costing twice p_x, or ii) $p_x = p_y$ can be prove simply assuming in the above proof that i) $p_x = \alpha$, $p_y = 2\alpha$ and $\alpha(X_1 + 2Y_1) + \gamma Z_1 = \kappa$, $\alpha(X_2 + 2Y_2) + \gamma Z_2 = \kappa_1$, ii) $p_x = p_y = \alpha$, $\alpha(X_1 + Y_1) + \gamma Z_1 = \kappa$, $\alpha(X_2 + Y_2) + \gamma Z_2 = \kappa_{1.}$

Proof of Proposition 3 (Case with equal prices)

(a) When we have equal prices the equations (12) become:

$$\begin{aligned} f_{y1} &= \frac{X_1 + Y_1 + Z_1}{2} = 1/2, \\ f_{x2} &= \frac{X_2 + Y_2 + Z_2}{2} = 1/2, \end{aligned}$$

which imply:

$$\begin{aligned} X_1 + Y_1 + Z_1 &= 1, \\ X_2 + Y_2 + Z_2 &= 1; \end{aligned}$$

then using the Lemma we also have $X_3 + Y_3 + Z_3 = 1$ and so equilibrium with prices all equal imply that the Hirota conditions hold.

(b) Conversely suppose the Hirota conditions hold. Then we have to show that this implies that $p_x = p_y = 1/3$. Again multiplying through (12), we get the linear system:

$$\begin{aligned} p_x X_1 + p_y Y_1 + (1 - p_x - p_y) Z_1 &= (1 - p_x)/2, \\ p_x X_2 + p_y Y_2 + (1 - p_x - p_y) Z_2 &= (1 - p_y)/2. \end{aligned}$$

Solving these linear equations we get:

$$\begin{aligned} p_y &= -\frac{(-\frac{1}{2}X_2 + Z_1 X_2 + \frac{1}{2}X_1 - Z_2 X_1 - \frac{1}{2}Z_1 + \frac{1}{4})}{(Y_2 X_1 - Y_2 Z_1 + \frac{1}{2}Y_2 - Z_2 X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1 Z_2 - Y_1 X_2 + Z_1 X_2)}, \\ p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2 Z_1 - \frac{1}{2}Y_1 + Y_1 Z_2)}{(Y_2 X_1 - Y_2 Z_1 + \frac{1}{2}Y_2 - Z_2 X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1 Z_2 - Y_1 X_2 + Z_1 X_2)}. \end{aligned}$$

This solution requires that the determinant condition

$$(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2) \neq 0$$

should hold.

Imposing the Hirota conditions

$$X_1 = 1 - Y_1 - Z_1; Y_2 := 1 - X_2 - Z_2$$

gives $p_x = 1/3; p_y = 1/3$. ■

Proof of Proposition 4 (Pareto optima Class II)

(a) In the equilibrium $u_1 = 1 - a, u_2 = u_3 = a$, which imply excess supply for good x. Thus $p_x = 0$. Suppose that $p_y = k$, the equilibrium condition for individual 1 and 2 are:

$$\begin{aligned} kY_1 + (1-k)Z_1 &= 1 - a, \\ \frac{(kY_2 + (1-k)Z_2)}{(1-k)} &= a, \end{aligned}$$

which imply:

$$\begin{aligned} kY_1 + (1-k)Z_1 &= 1 - a, \\ kY_2 + (1-k)Z_2 &= (1-k)a. \end{aligned}$$

(b) Suppose that (8) hold. We have to show that $p_y = k$. The equilibrium condition for individual 1 is:

$$p_yY_1 + (1 - p_y)Z_1 = 1 - a$$

and

$$p_y = \frac{(a + Z_1 - 1)}{(-Z_1 + Y_1)}.$$

Substituting (8) of individual 1,

$$Y_1 = ((1-a) - (1-k)Z_1)/k,$$

we get:

$$p_y = k.$$

Proposition 8 can be similarly proved imposing that $p_y = 1/2$.

B Stability of Equilibrium with Equal Utilities

B.1 Unequal Prices

The determinant of (11) when $p_x = \alpha$ and $p_y = \beta$ is equal to

$$d = \frac{((1-2Z_2)(1-2Y_1) + 2Y_2(1-2Z_1))(1-\alpha-\beta)}{2\alpha(1-\alpha)(\alpha+\beta)(1-\beta)} \tag{14}$$

whose sign is given by that of $(1-2Z_2)(1-2Y_1) + 2Y_2(1-2Z_1)$.

The trace is equal to

$$\begin{aligned} t \quad = \quad & -\frac{(2\beta\alpha^2 + \beta^2 - 2\alpha^2 - \beta - \beta\alpha + \alpha)}{a(1-a)(a+\beta)(1-\beta)} Y_1 \\ & -\frac{(\beta\alpha + 2\beta^2 - \alpha^2 - 2\beta^2\alpha - \beta + \alpha)}{a(1-a)(a+\beta)(1-\beta)} Y_2 \\ & -\frac{(2\beta^2\alpha - 2\beta^2 - 1 + \alpha + 3\beta - 3\beta\alpha)}{a(1-a)(a+\beta)(1-\beta)} Z_2 \\ & -\frac{(-2\beta\alpha^2 - \beta^2 + 2\alpha^2 - 1 + 2\beta)}{a(1-a)(a+b)(1-b)} Z_1 \\ & -\frac{(1-\alpha+2\beta\alpha+\beta^2-2\beta-\beta^2\alpha)}{\alpha(1-\alpha)(\alpha+\beta)(1-\beta)}, \end{aligned} \tag{15}$$

which we can write as

$$\begin{aligned} t \quad = \quad & \frac{Y_1(\beta-\alpha+2\alpha^2)}{(\alpha+\beta)(1-\alpha)\alpha} + \frac{Y_2(\beta-\alpha-2\beta^2)}{(1-\beta)(\alpha+\beta)\alpha} \\ & -\frac{(2\beta-1)Z_2}{(\alpha+\beta)\alpha} + \frac{Z_1(1-\beta-2\alpha^2)}{(\alpha+\beta)(1-\alpha)\alpha} - \frac{(1-\beta)}{(\alpha+\beta)\alpha}. \end{aligned}$$

This is of ambiguous sign.

In the case in which p_y costing twice p_x, the stability conditions in terms of determinant and trace are:

$$d = \frac{(1+4Y_1Z_2 - 2Y_1 - 4Y_2Z_1 - 2Z_2 + 2Y_2)(1-3\alpha)}{6\alpha^2(1-\alpha)(1-2\alpha)},$$

$$
\begin{aligned}
t = & \frac{Y_1(\alpha + 2\alpha^2)}{(1-\alpha)3\alpha^2} + \frac{Y_2(\alpha - 8\alpha^2)}{(1-2\alpha)3\alpha^2} \\
& - \frac{(4\alpha - 1)Z_2}{3\alpha^2} + \frac{Z_1(1 - 2\alpha - 2a^2)}{(1-\alpha)3\alpha} - \frac{(1-2\alpha)}{3\alpha^2}.
\end{aligned}
$$

Still the sign is ambiguous.

In the case in which $p_x = p_y$, then (14) and (15) become respectively:

$$
d = \frac{[(1-2Z_2)(1-2Y_1) + 2Y_2(1-2Z_1)](1-2\alpha)}{4\alpha^2(\alpha-1)^2},
$$

$$
t = \frac{(Y_1 - Y_2)}{(1-\alpha)} - \frac{Z_2(-1+2\alpha)}{2\alpha^2} + \frac{Z_1(1-2\alpha)(\alpha+1)}{2\alpha^2(1-a)} - \frac{(1-\alpha)}{2\alpha^2}.
$$

There are special cases of $p_x = p_y$:

1) *Equal endowments of each good for individuals* 1 and 2 ($X_1 = X_2$, $Y_1 = Y_2$, $Z_2 = Z_1$). This case is always stable.

Note that with equal endowments for the first two individuals, each endowment is at most equal to 1/2, i.e. $X_1 \leq 1/2$, $Y_1 \leq 1/2$, $Z_1 \leq 1/2$. The determinant is:

$$
d = \frac{(2Z_2 - 1)(2\alpha - 1)}{4\alpha^2(\alpha - 1)^2},
$$

which is always positive since $(2\alpha - 1) < 0$ and $(2Z_2 - 1) < 0$.

The trace becomes:

$$
\begin{aligned}
t &= \frac{4Z_2\alpha - 2\alpha + \alpha^2 - 2Z_2 + 1}{2\alpha^2(\alpha - 1)} \\
&= \frac{\alpha^2 + (1 - 2Z_2)(1 - 2\alpha)}{2\alpha^2(\alpha - 1)},
\end{aligned}
$$

that is always negative since $(\alpha - 1) < 0, 1 - 2Z_2 > 0$ and $1 - 2\alpha > 0$.

2) *Equal endowments of* Z *for individuals* 1 and 2 ($Z_1 = Z_2$):

$$
d = \frac{-(1-2Z_2)(1-2(Y_1 - Y_2))(-1+2\alpha)}{4\alpha^2(-1+\alpha)^2}.
$$

Since $(2\alpha - 1) < 1$ and $1 - 2Z_2 > 0$, the determinant is positive if $(1 - 2(Y_1 - Y_2) > 0$. This holds if $Y_2 > Y_1$ or if $Y_1 < 1/2$. The trace is:

$$t = \frac{(Y_1 - Y_2)}{(1-\alpha)} + \frac{(2\alpha - 1)Z_2}{\alpha^2(\alpha - 1)} - \frac{(1-\alpha)}{2\alpha^2},$$

which is certainly negative when the conditions that make the determinant positive hold.

B.2 Three equal prices

In this case, $Z_1 = 1 - Y_1 - X_1; Z_2 = 1 - Y_2 - X_2$ and $a = b = 1/3$. The determinant is:

$$d = \frac{27}{14}(-\frac{1}{4} + \frac{1}{2}Y_1 + \frac{1}{2}X_2 - X_2Y_1 + Y_2X_1),$$

which is positive if:

$$X_1Y_2 > (\frac{1}{2} - X_2)(\frac{1}{2} - Y_1). \tag{16}$$

The trace is:

$$t = \frac{3}{2}(-Y_1 - 2Y_2 + 1 - X_2 - 2X_1).$$

The trace is negative if:

$$-Y_1 - 2Y_2 + 1 - X_2 - 2X_1 < 0. \tag{17}$$

We can combine (17) and (16) to derive an endowment restriction which is necessary and sufficient for local stability of this case: we require

$$X_2 > \max\{1 - 2Y_1 - Y_2 - 2X_1, \frac{(2Y_1 - 1/2)}{2X_1 + 1. - 2Y_1}\}.$$

If this fails we can for example have the determinant and trace both positive. For example take $Y_1 = Y_2 = 1$ and $X_1 = .05, X_2 = .45$. Then (16) is .0025 and (17) is 0.15.

C Stability with unequal utilities (Class II)

C.1 Stability for any initial conditions starting with $p_x = 0$

In the more general case with equilibrium prices $p_y = k, p_z = 1-k$, the endowment becomes:

$$\begin{aligned} Y_2 &= ((1-k)/k)(a-Z_2); Z_3 = (k/(1-k))(a-Y_3), \\ Y_1 &= 1/k[1-a-(1-k)Z_1], \\ Z_1 &= 1-Z_2-Z_3, \end{aligned}$$

$$E_y = \frac{(p_y(1-k)(a-Z_2))/k + (1-p_y)Z_2)}{(1-p_y)} + \frac{p_y Y_3 + \frac{(1-p_y)(a-Y_3)k}{1-k}}{p_y} - 1.$$

Computing its derivative and evaluating at $p_y = k$ we obtain

$$\frac{\partial E_y}{\partial p_y} = \frac{-2a + Y_3 + Z_2}{k} < 0. \tag{18}$$

The equilibrium is always stable since $(Y_3 - a) < 0$ and $(Z_2 - a) < 0$.

As an example we know that a Pareto optimum with unequal utilities can be supported as an equilibrium with two equal prices $p_y = p_z = 1/2$ when the endowment distribution is:

$$Y_1 = 2 - 2a - Z1, Y_2 = a - Z_2; Y_3 = a - Z_3.$$

With this endowment distribution, the excess demand function for y has the form

$$E_y = p_y(2-2a-Z_1) + (1-p_y)Z1 + \frac{p_y(a-Z_3) + (1-p_y)Z_3}{p_y} - 1.$$

Then we obtain:

$$\frac{\partial E_y}{\partial p_y} = 2 - 2a - 2Z_1 - \frac{Z_3}{p_y^2}.$$

We should evaluate it at the equilibrium: $p_y = 1/2$, obtaining:

$$\frac{\partial E_y}{\partial p_y} = 2 - 2a - 2Z_1 - 4Z_3 < 0. \tag{19}$$

Note that $2(1-Z_1-Z_3) = 2Z_2$. Thus (19) becomes: $2(-a+Z_2-Z_3) < 0$ since $Z_2 - a < 0$ (Y_2 cannot be negative).

C.2 Stability for arbitrary initial conditions

The determinant of (11) when $p_x = k$, $p_y = 1-k$ and $Y_1 = (-(1-k)Z_1 + (1-a))/k$; $Y_2 = (-(1-k)Z_2 + (1-k)a)/k$ is

$$d = \frac{a[2(X_1+X_2)(1-k)+k-2(1+Z_2)]+2a^2-2(1-k)[(Z_2X_1-Z_1X_2+X_2)]+(1-Z_1+Z_2)])}{k^2(1-k)}.$$

The trace is equal to

$$t = \frac{k^2(2a-1+2X_2-2Z_2+X_1)+k(2(1-a)-Z_1-X_2-X_1))-1+Z_1+Z_2}{k^2(1-k)}.$$

The sign of the trace and of the determinant are ambiguous.